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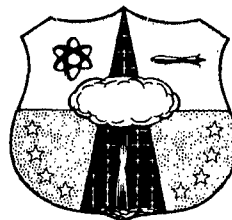
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THE INCLUSION OF MATERIAL STRENGTH
IN HYDRODYNAMIC CALCULATIONS

April 1963

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FOREWORD

This report was prepared by the Aeroelastic and Structures Research Laboratory, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology. It is designated the Aeroelastic and Structures Research Laboratory Technical Report 106-1. The work reported was performed under USAF Contract AF 29(601)-4601, and was administered by the Air Force Special Weapons Center, Kirtland Air Force Base, New Mexico. Lt. J. Rich was the project engineer for AFSWC.

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ABSTRACT

This is an Interim Report discussing the development of a theory for describing the response of solid bodies to high intensity impulsive loads, including the effect of material strength.

Inclusion of material strength requires a full thermodynamic treatment, which has not been satisfactorily formulated. An approximation is considered in which entropy production due to plastic flow is neglected. In the resultant theory, stresses are related to elastic strains through an isentropic strain energy potential, and the elastic strains are limited by a yield condition. Results of an analysis of dynamic compressibility data for copper to 2.7 Mbar by second order elasticity theory are very encouraging. Analysis of experimental plane wave propagation data for aluminum with a variety of yield functions indicates that presently measurable quantities do not provide a sensitive means of determining the yield function. Other configurations may be more suitable, but require development of two-dimensional solution methods.

PUBLICATION REVIEW

This technical documentary report has been reviewed and is approved.


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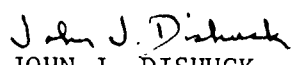

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LIST OF SYMBOLS

a	Lagrangian coordinate
g	Rate function
h	Heat flux
l	Elastic constant
m	Elastic constant
n	Elastic constant
p	Pressure
q	Heat supply
s	Entropy
t	Time
u	Displacement
x	Eulerian coordinate
E	Young's modulus
F	Force
G	Shear modulus
K	Bulk modulus
T	Temperature
ϵ	Strain
λ	Lamé constant
μ	Lamé constant
ν	Substate
ρ	Density
σ	Yield stress
τ	Stress
ψ	Free energy
\mathcal{E}	Internal energy

The Cartesian tensor notation is used throughout.
Thus lower case subscripts denote tensor indices and take

on values 1, 2, 3, respectively. Repetition of indices in a term implies summation. The comma denotes partial differentiation with respect to the space variables, while the dot is used to denote the material derivative.

Greek subscripts or superscripts are not tensorial indices and take on values 1, 2, . . . M where M is not necessarily 3. However, the summation rule will be also applied to Greek indices. Since Greek indices do not denote tensorial character, they will be written as subscripts or superscripts according to convenience.

For a given set of tensor components $A_{i \dots j}$, we will occasionally write \underline{A} , which may be read as "boldface A." Similarly we will use \underline{B} to denote the set B^α .

The usual symbols for a symmetric $A_{(ij)}$ and an antisymmetric $A_{[ij]}$ tensor will be used.

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1. INTRODUCTION

A number of engineering problems involving very high velocity impact, or other very rapid modes of deposition of large amounts of energy in a structure have recently become important.

The first approach to such problems has involved the so-called "hydrodynamic" assumption. It is assumed that the shear stress which the material is able to support is always negligible in comparison with the compressive stress, and the material is treated as if it were a fluid. A suitable "equation of state" is used, which is deduced from plate impact experiments via the Mie-Grüneisen theory,¹ and at higher pressures is deduced from the Fermi-Thomas-Dirac theory.² While the "hydrodynamic" theory has been found to give reasonably good results for many problems, it is not able to account for important effects which are traceable to the influence of material strength.

It is therefore necessary to inquire if material strength can be included in the theory. The kinematical equations of continuity and motion, expressing the principles of conservation of mass and momentum, are well known. To these must be joined constitutive relations expressing the reaction of the material. Two important problems arise.

The first problem centers on the question whether solutions can be found for this set of nonlinear differential equations for specific initial and boundary conditions. The question must await formulation of specific constitutive re-

lations, but the possibility of obtaining analytical solutions to nontrivial problems seems remote. There remains the possibility of numerical integration, already widely used for "hydrodynamic" calculations. Such a method of solution necessarily cannot provide fine detail due to limitations on computer storage and running time and accumulation of truncation and round-off errors. Again, the question whether sufficient detail can be reproduced so that the solution is an improvement over a "hydrodynamic" solution cannot be fully assessed at this time.

The second problem centers on the specification of the constitutive relations. Constitutive relations have been formulated for certain ideal materials; e.g., viscous fluid, linear elastic solid, perfect plastic solid, etc. It is not immediately evident how the constitutive relations are to be formulated for the present case. Only for specially simple ideal materials do the constitutive relations involve the kinematical variables, stress and strain, alone. It is necessary, in general, to consider also thermodynamic variables and the interaction between thermodynamic and kinematic changes. A satisfactory general thermodynamic treatment for a solid has so far not been given.

The problem therefore is not so much one of measurement of material properties for insertion in the constitutive relations. It is first necessary to achieve a satisfactory formulation of the constitutive relations and to specify which physical properties are relevant. There is a fundamental aspect of the theory of continua which renders this difficult. The constitutive relations must describe the behavior of a volume element of the material. This behavior cannot be deduced from gross measurements on a body of the material without recourse to the theory itself.

Thus, in order to deduce stress-strain relations from measurements of elongations produced by surface tractions applied to a body of the material under study, it is necessary to analyze the deformation field correctly. For a tensile specimen, the deformation field may be assumed to be particularly simple for small elongations, so that the stress-strain relation can be deduced directly from the tensile force-elongation data. As soon as the specimen necks, this can no longer be done. When a tension test is conducted at high rates, it would, in addition, be necessary to consider thermodynamic changes explicitly. Failure to analyze the experiments correctly probably accounts for apparent contradictions in the dynamic "material properties" reported by various experimenters.

The present paper is an interim report on an investigation of the possibility of correcting the "hydrodynamic" theory to include the gross observable effects of material strength. Some pertinent thermodynamic relations are reviewed. In view of the fact that the thermodynamic treatment has not been completed, some interim approximations are suggested which depend on the working hypothesis that the yield stress is not greatly increased above its static value, and which permit uncoupling of kinematic and thermodynamic considerations, leading to a fairly simple formulation of the constitutive relation. A beginning is then made in the consideration of two aspects of the resultant theory; viz., the relation between stress and the recoverable (elastic) part of the strain, and the yield condition which in effect limits the recoverable part of the strain.

2. SUMMARY

2.1 Kinematical Relations

The Eulerian equations of continuity and motion are well known

$$\dot{\rho} + \rho \dot{u}_{i,j} = 0 \quad (3.2)^*$$

$$\tau_{ij,j} = \rho(\ddot{u}_i - F_i) \quad (3.3)$$

To these must be joined constitutive equations in the general form

$$f(\tau_{ij}, u_{i,j}, \dots) = 0 \quad (3.4)$$

expressing the behavior of the material, before proceeding to solutions of problems with suitable initial and boundary conditions. The possibility of finding closed form solutions to these equations, which contain both geometrical and physical nonlinearities is remote. For special cases, the method of characteristics may prove useful, but a general discussion cannot be given until a specific constitutive relation is inserted.

There is a possibility that numerical integration of the equations in finite difference form may provide solutions sufficient for engineering purposes for some problems. The principal difficulty in the construction of finite difference analogs to Eqs. 3.2, 3.3 and 3.4 is the representation of stress and deformation gradients. Schemes in which

*Equation numbers in this section are the same as those in the main body of the text. For definition of symbols see main text. For notation, see note at end of the List of Symbols.

averages over nearest neighbors are used, which are employed in Eulerian fluid flow problems, could probably be extended to the present case. Alternatively, the Lagrangian equations, in which the difficulty does not arise, could be used for motions involving moderate distortions. A finite difference code could be programmed, in which the constitutive relation appears as a subprogram, resulting in considerable flexibility. The constitutive relation could then be easily altered without disturbing the control program or calculations involving the equations of continuity and motion.

2.2 Constitutive Relation

A discussion of the constitutive relation necessarily requires consideration of thermodynamic relationships. Only in very special circumstances can the state of the material be described in terms of kinematical variables alone. The main thermodynamical results are reviewed. In particular, the principle of conservation of energy leads to the relation that the increase in internal energy is due to the external stress power, P_E and the external heat addition Q_E ,

$$\rho \dot{\mathcal{E}} = P_E + Q_E \quad (4.2)$$

Definition of entropy leads to an equation for entropy production, which is found to be due to the difference between the external stress power and the internal stress power P_I , i.e., to the dissipative stress power, and to the external heat addition

$$\rho T \dot{s} = P_E - P_I + Q_E \quad (4.13)$$

The complete set of equations to be considered are then the equation of continuity and motion, Eqs. 3.2, 3.3,

the energy and entropy equations Eq. 4.2, 4.13 together with a caloric equation of state governing material behavior

$$\mathcal{E} = \mathcal{E}(s, \underline{v}) \quad (4.5)$$

where \underline{v} determine the mechanical substate, and phenomenological relations governing dissipative and heat-flow mechanisms.

Under special circumstances, such as when the motion is entirely adiabatic or isothermal, some or all of the thermodynamic quantities can be eliminated from explicit consideration, and the material can be adequately described by a simple stress-strain relation. For this reason, we prefer to include the thermodynamic relations as part of the constitutive relations.

One assumption which leads to such simplification is the "hydrodynamic" assumption, i.e., the yield stress is always small compared to the pressure. With the further assumption that spherical stress work is nondissipative and spherical strain is entirely recoverable, this leads to the result that the dissipative work done is negligible compared to the spherical stress work. For an adiabatic motion ($Q_E = 0$), we see from Eq. 4.13 that $\dot{s} = 0$ while Eq. 4.2 becomes simply

$$\rho \dot{\mathcal{E}} = p \dot{\rho} / \rho \quad (4.36)$$

i.e., the energy equation for an ideal fluid. Entropy changes at possible shock waves may be handled by the Rankine-Hugoniot relations, or by the introduction of the usual type of artificial viscosity smoothing function which is so formulated that its effect is negligible everywhere except in areas of very high gradient, such as in shock zones.

As soon as the yield stress cannot be assumed small in some regions of the motion, the "hydrodynamic" assumption is violated, and it is necessary to consider a full thermodynamic treatment of the motion. However, a fully satisfactory thermodynamic treatment has so far not been given. An interim formulation, in which the entropy production due to dissipative plastic flow is still neglected may be considered until a thermodynamic treatment becomes available. While this may provide some correction to the "hydrodynamic" theory, the basic assumption ($\dot{s} = 0$) is violated most seriously in just those areas where the effect of material strength is likely to be greatest, i.e., areas where a large amount of plastic flow occurs at low pressure. The energy equation is modified simply to

$$\rho \dot{\mathcal{E}} = \tau_{ij} \dot{u}_{i,j}^e \quad (4.39)$$

where \underline{u}^e is the elastic part of the deformation, which is limited, in effect, by a yield condition.

2.3 Elastic Behavior

We turn now to preliminary considerations of the elastic stress-strain relationship Eq. 4.39, and of the yield condition.

Following usual practice, the strain energy potential $\rho \mathcal{E} = \Psi(\underline{\epsilon})$, where the strain $\underline{\epsilon}$ is defined by

$$\epsilon_{kj} = \frac{1}{2} (u_{k,j} + u_{j,k} - u_{i,j} u_{i,k}) \quad (5.1)$$

is expanded as a power series in the strain invariant I_1, I_2, I_3

$$\Psi = aI_1 + bI_1^2 + cI_2 + dI_1^3 + mI_1I_2 + nI_3 + \dots \quad (5.2)$$

So that the theory agrees with the classical theory for infinitesimal strains, we set $a = 0$, $b = 1/2 \times (\lambda + 2\mu)$, $c = -2\mu$ where λ and μ are the adiabatic Lamé constants.

The stress-strain relation is then obtained from

$$\tau_{ij} = (b_{ik} - 2\epsilon_{ik}) \left(\frac{\partial \Psi}{\partial \epsilon_{kj}} \right)_s \quad (5.3)$$

To complete the description, the material constants λ , μ , ℓ , m and n must be evaluated for a particular value of entropy. A parallel development exists for isothermal conditions, when the strain energy potential is equal to the free energy $\rho\psi = \Psi(\underline{\epsilon})$. The constants λ , μ , ℓ , m and n will then refer to a particular value of temperature.

The first-order coefficients, the adiabatic Lamé constants, are best determined by ultrasonic pulse-echo techniques. Such experiments can also be performed under hydrostatic pressure, although the attainable pressures are limited to about 10 kb by present techniques, from which pressure derivatives of the elastic moduli can be determined. For an isotropic material under hydrostatic pressure, only two distinct wave velocities can be measured, the longitudinal wave velocity, corresponding to $\sqrt{(K + 4G/3)/\rho}$ and the transverse wave velocity corresponding to $\sqrt{G/\rho}$, where K is the Bulk Modulus, and G is the Shear Modulus.

By considering a state of strain consisting of a (large) spherical strain e , and an infinitesimal perturbation e_{rs}

$$\epsilon_{rs} = e\delta_{rs} + e_{rs} \quad (5.9)$$

it is possible to evaluate the effect of strain on the

Bulk Modulus and Shear Modulus

$$G = \frac{1}{2}(1 - 2e)^{5/2} \left\{ 2\mu - e(6\lambda + 8\mu - 3m + n) - e^2(54\ell + 12m) \right\} \quad (5.14)$$

$$K = \frac{1}{3}(1 - 2e)^{5/2} \left\{ (3\lambda + 2\mu) - e(21\lambda + 14\mu - 54\ell - 18m - 2n) - e^2(243\ell + 81m + 9n) \right\} \quad (5.15)$$

Thus, measurements of pressure derivatives of the elastic moduli should provide two independent relations for the three second-order constants ℓ , m , and n ; but this is not sufficient to determine these constants separately, and a further independent measurement must be sought.

Information relevant to the second-order elastic constants may also be deduced from adiabatic compressibility data under spherical stress. For a purely spherical stress, we find

$$p = \frac{1}{2} \left\{ \left(\frac{\rho}{\rho_0} \right)^{7/3} - \left(\frac{\rho}{\rho_0} \right)^{5/3} \right\} \left\{ 3\lambda + 2\mu \right\} + \frac{1}{2}(27\ell + 9m + n) - \frac{1}{2}(27\ell + 9m + n) \left(\frac{\rho}{\rho_0} \right)^{2/3} \quad (5.8)$$

Adiabatic and isothermal compressibility data may be deduced from hydrostatic compression experiments at pressures up to 100 kb, and from dynamic Hugoniot measurements from plate impact experiments up to 4 Mb, via the Mie-Grüneisen theory. If sufficient terms are included in the

expansion, Eq. 5.2, then from Eq. 5.8 we see that a plot of

$$\frac{p}{(\rho/\rho_0)^{7/3} - (\rho/\rho_0)^{5/3}} \text{ vs } (\rho/\rho_0)^{2/3}$$

should be a straight line. This is in fact found to be so for copper (figures 5.1, 5.2), the only material for which the comparison has so far been made.

Furthermore, the straight-line fit yields a value of $(27\ell + 9m + n)$. For the zero temperature isotherm, this is found to be -5880 kb, for the isentrope through $P = 0$, $T = 293^\circ\text{K}$, we find a value -5840 kb, while for the isentrope through 500 kb on the hugoniot we find a value -4460 kb.

The experiment of Altshuler³⁵ gives additional information. In this experiment, the speed of a relaxation wave propagating into shock compressed material is determined. The "overtaking-relaxation method" yielded the speed of the bulk wave which was found to correspond closely to $\sqrt{K/P}$ determined from adiabatic compressibility data estimated from slopes of measured shock Hugoniots. However, the "lateral relaxation method" led to velocities which were higher than the bulk wave velocity and corresponded to the longitudinal wave velocity. The measured velocity of a longitudinal release wave moving into copper shock compressed to 407 kb was found to be 6.33 Km/sec, yielding $(K + 4G/3) = 4290$ kb. Rough interpolation of the bulk modulus for these conditions gives $K = 3160$ kb, so that we have, very approximately, $G = 850$ kb, which is nearly twice the value at zero pressure.

A comparison with the information deduced from pressure derivatives of the elastic moduli measured by pulse-echo techniques should be possible, but this calculation has not yet been performed.

2.4 Yield Behavior

We turn now to investigate certain aspects of the yield condition relevant to the present problem. There is a large number of effects which may alter the yield condition. Among these one might mention temperature, strain-rate, strain hardening, and, under extreme conditions, the yield stress may also conceivably depend on the pressure. It would probably be undesirable, even if it were possible, to incorporate a physically realistic yield dependence on all of the relevant factors. Rather, we seek a rough description which will be adequate for engineering calculations and which will reflect only the most important gross effects.

It is not yet clear which effects are the most important under conditions of high pressure and strain rate, and many contradictory statements appear in the literature with regard to the magnitude of one or the other of the above-mentioned effects.

We therefore choose the simplest possible heuristic approach to investigate the effects of various types of postulated behavior on observable response and seek to compare the predicted response with experimental observations.

The yield behavior is best observed at low pressure and it is sufficient initially to limit the discussion to infinitesimal strain. A generalization of the Malvern theory³⁶ gives the stress-strain relation

$$\dot{\tau}_{ij} = 2G \dot{\epsilon}_{ij} + (K - 2G/3) \delta_{ij} \dot{\epsilon}_{kk} - 2G g(\bar{\tau}^*) \frac{\tau'_{ij}}{\bar{\tau}} \quad (6.11)$$

Here $g(\bar{\tau}^*)$ is a strain-rate function, where $\bar{\tau}^*$, the von Mises effective overstress, is defined by $\bar{\tau}^* = \bar{\tau} - \bar{\sigma}$, and

$$\bar{\tau} = \sqrt{\frac{1}{2} \{ (\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2 \}} \quad (6.7)$$

Correspondingly, $\bar{\sigma}$ is formed from the stress $\underline{\sigma}$ which satisfies the yield condition.

We therefore need to evaluate the yield stress $\underline{\sigma}$ and the relaxation function g in order to complete the description of the material.

It is difficult to decide on the functional form of the relaxation function g . The best that can be done at present is to choose a simple functional form for g which is consistent with current theories of the physical processes taking place and then to use experimental results to put bounds on the values of the coefficients. With the tenuous justification that plastic flow is due to propagation of dislocations and is a thermally activated process, the relaxation function is given a form

$$g(\bar{\tau}^*) = a \bar{\tau}^* e^{b\bar{\tau}^*} \quad (6.31)$$

where a and b are to be found empirically.

The yield condition is also difficult to evaluate. It is quite likely that the yield condition may differ from that applicable to "static" conditions. In the time taken for static tests there may be, for example, ageing and self-annealing mechanisms active which are inoperative in the time available under dynamic conditions. Bell and Werner⁴¹ have found, for example, that the stress-strain curve deduced from a rate-independent analysis of wave propagation in rods differs from the stress-strain curve obtained in a static tensile test.

The information on yield stress applicable to "static" conditions cannot be taken over directly for dynamic problems; it is again necessary to assume a yield behavior and to evaluate the constants empirically.

Some simple descriptions of yield and relaxation behavior are investigated for uniaxial strain configurations. Four different types of behavior are considered: (1) rate independent with constant yield, (2) rate independent with pressure dependent yield, (3) rate independent with strain hardening, and (4) constant yield with strain-rate behavior. The results indicate that it is not possible to distinguish behavior from measurable quantities; rear surface motion and stress or interface stress, for the assumed range in variables and uniaxial strain conditions. It is desirable to extend the investigation to higher pressures and to other configurations, but methods of solution for such cases must be developed before the experimental results could be interpreted.

2.5 Conclusion

It is necessary that each of the approaches considered here be pursued further. In particular it is urgently necessary to pursue the thermodynamic treatment of the material, since the proposed approximations are suspect.

The agreement obtained between compressibility data and second-order elasticity theory is encouraging, and indicates that further investigation is warranted. The elastic description has not been completed. In particular another independent measurement of the second-order elastic constants must be sought.

Description of the yield behavior is far from complete. It appears that the simple rate-independent elastic-plastic theory is in accord with what little experimental evidence exists for uniaxial strain configurations at low stresses, but there is reason to believe that this will not be so for other configurations. Since solutions are obtainable only for uniaxial strain at the present time, further progress must await development of solution methods for more complex configurations.

3. KINEMATICAL RELATIONS

3.1 Equations of Motion

We consider a body initially in a state B_0 . The coordinates of a particle P referred to a Cartesian common frame are denoted by a_i . At time t , the body is in a distorted state B , and the coordinates of the particle P referred to the common frame are now x_i , related to the a_i by the reversible transformation law governing the motion.

$$x_i = x_i(\underline{a}, t) \quad a_i = a_i(\underline{x}, t) \quad (3.1)$$

For our present purposes it is not necessary to consider general curvilinear coordinate systems, and to pursue the argument in fully invariant form.

The displacement of the point P is given by $u_i = x_i - a_i$. The velocity of the point P is given by $\dot{x}_i = \dot{u}_i$.

We limit the discussion to a nonpolar, isotropic, homogeneous medium.

Taking the Eulerian viewpoint, by observing a volume element with sides dx_i with particles flowing through it, and taking a_i as a function of x_i , Eq. 3.1₂, the equation of continuity expressing the principle of conservation of mass is³

$$\dot{\rho} + \rho \dot{u}_{i,j} = 0 \quad (3.2)$$

while the equation of motion, expressing the principle of conservation of momentum becomes

$$\tau_{ij,j} = \rho(\ddot{u}_i - F_i) \quad (3.3)$$

where $\underline{\tau}$ is the symmetric Eulerian stress tensor, ρ the density, and \underline{F} the specific body force.

Alternatively, the Lagrangian viewpoint may be taken, by observing a volume element with sides da_i containing a fixed collection of particles, and taking x_i as a function of a_i , Eq. 3.1₁. However, it is then necessary to refer both body force and stress tensor to the original state of the body B_0 . The Lagrangian force and stress tensors thus lose direct physical meaning. Moreover, the Lagrangian stress tensor is not symmetrical. This can be remedied by using the Piola-Kirchoff stress tensor, at the cost of increasing the complexity of the equation of motion. For present purposes the Lagrangian viewpoint is not helpful, and will not be pursued here.

The equations, Eq. 3.2, 3.3, cannot be integrated until constitutive equations are supplied, which describe the reaction of the material to the deformation. These relate the stress and deformation tensors, and take the general form

$$f(\tau_{ij}, u_{i,j} \dots) = 0 \quad (3.4)$$

and will be considered in detail in Section 4.

3.2 Characteristics Method of Solution

In certain specially simple cases, the equations 3.2, 3.3 may be reduced to a system of μ quasi-linear equations in μ unknown dependent functions ϕ^μ , of ν variables x^ν

$$B_{\mu\eta}^\nu \phi_{,\nu}^\mu + D_\eta = 0 \quad \eta = 1, \dots, \mu \quad (3.5)$$

where B and D are functions of ϕ^μ and x^ν . If these equa-

tions are totally hyperbolic⁴ then there exist $(v - 1)$ dimensional characteristic surfaces through any point x^v such that linear combinations of the equation 3.5 involve derivatives of ϕ^μ only in directions lying in these surfaces. This results in a decisive simplification only under special circumstances, i.e., when these derivatives are all in the same direction, or when the number of independent variables v is equal to 2.

For a solid which can support a shear stress, only the latter case has been explored, e.g., uniaxial motion of a solid with a purely mechanical elastic-plastic stress-strain relation.^{5,6} Even in this case, the characteristic method is unwieldy except for specially simple boundary conditions.

It is not possible to pursue the argument further at this point for the case $v > 2$, until a specific constitutive equation has been inserted, since reduction of Eqs. 3.2, 3.3, 3.4 to the form Eq. 3.5, and the conditions under which Eqs. 3.5 are totally hyperbolic depend on the specific form of the constitutive relation.

3.3 Finite Difference Method of Solution

One method of solution of the initial value problem posed by Eqs. 3.2, 3.3 and 3.4 together with suitable initial and boundary conditions is to replace the derivatives by suitable finite difference approximations, and to carry out the integration numerically. The technique has been used extensively for the "hydrodynamic" problem.⁷

As with fluid flow problems, care will have to be taken to ensure that the solution remains continuous everywhere, if necessary by the introduction of suitable smoothing functions into the constitutive relation.⁸

In the finite difference method, the continuum is represented by a finite number of points at distances Δx_i apart, and the continuous solution is represented by a series of average values of the dependent variables over the intervals Δx_i , Δt . The original state B_0 of the body is thus represented by a finite number of discrete points. The coordinates of a discrete point P in B_0 are $a_i = (\eta\Delta x_1, \xi\Delta x_2, \zeta\Delta x_3)$ where η, ξ, ζ are integers.

The deformed state of the body B is also represented by the same number of discrete points, with coordinates of P in B , x_i , being given by the transformation law Eq. 3.1. These coordinates do not in general coincide with integral multiples of Δx_i .

The main difficulty in constructing an Eulerian finite difference scheme consists in obtaining suitable finite difference representations of the displacement and stress gradients in Eqs. 3.2, 3.3. Several ingenious schemes have been developed for the case of fluids, e.g., Harlow's Particle and Force code⁹ and Kolsky's nearest neighbor code.¹⁰ In these codes the nearest neighbors of the particle currently being considered are used in an averaging scheme to deduce the required gradients. Some such scheme could undoubtedly be used in the present instance also. There are other well-known difficulties associated with the Eulerian finite difference method. One of these is that boundaries between different materials are subject to diffusion.

Both of the above difficulties are avoided in the Lagrangian representation. Gradients are taken with respect to a_i in the initial configuration, and particles are thus always in the appropriate positions for constructing finite difference approximations. The stress tensor must be referred to the initial configuration, but this does not pose

any special problem. However, the difficulties associated with severe distortions of the Lagrangian mesh are well known.¹¹ For moderate distortions, however, the formulation of Lagrangian finite difference equations should not pose any serious problem.

The feasibility of carrying out specific calculations on present computers in terms of storage and machine time cannot be adequately assessed until the finite difference equations are formulated, and a specific problem is considered. However, some general remarks are possible. We consider the two-dimensional case (rectangular Cartesian or cylindrical polar coordinates).

The essential complexities introduced when material strength is included are the introduction of three stress components (τ_{11} , τ_{12} , τ_{22}) in place of the scalar pressure, three deformation gradients ($u_{1,1}$, $u_{(1,2)}$, $u_{2,2}$) in place of the scalar density, and the introduction of three constitutive equations in the place of one. The latter are also somewhat more involved, due to the necessity for a yield condition (discussed in the following section) and the fact that the constitutive equations are expressed in terms of the deformation gradients.

In a fluid calculation, it is generally necessary to store at least eight quantities at each mesh point; for example position r , z , velocities u , v , pressure p , density ρ , internal energy \mathcal{E} , and a parameter identifying the particle, i.e., mass of the particle m . In the present case, only four kinematical variables are added, two additional stress components and two additional deformation components. In addition there may be one or two extra thermodynamic variables.

The time required for computation of the equations of continuity, motion and constitutive relations will cer-

tainly be increased. However, a part of the total computation time is consumed in logic necessary to advance the calculation. Thus it seems reasonable to suppose that both computer storage and running time requirements will not be increased by much more than a factor of 1.5 to 2 for a problem in which material strength is introduced, over an equivalent fluid problem. Such an increase should not be prohibitive.

It might be argued that when material strength is introduced, a much finer mesh size will be required to resolve the fine detail of elastic-plastic wave interactions. If such resolution is required, then this is certainly true. However, numerous calculations have been performed using a one-dimensional finite difference code⁶ in which material strength was introduced, and various mesh sizes were used. Surprisingly, when extremely coarse mesh sizes were used, stress profiles still approximated the exact solution closely, although much of the fine detail was obliterated by the "smearing" commonly present in finite difference solutions. Moreover, displacements, which are integrated quantities, were reproduced very well.

The finite difference method thus removes much of the fine detail of the numerous wave interactions which makes computation of elastic-plastic problems by characteristic methods so laborious, while preserving nearly correct average behavior, the degree of "filtering" of fine detail depending on the coarseness of the mesh size. It is expected that a two-dimensional code would behave similarly. The mesh size would thus depend on the degree of detail required in the solution. For some purposes, this may not require a decrease in mesh size over that suitable for an equivalent fluid calculation.

It would thus appear from these preliminary considerations that it is feasible to write a finite difference code, particularly using the Lagrangian viewpoint, and to carry out useful calculations on presently available computers. Detailed estimates of storage and computation time requirements are impossible until such a code is formulated, and some experience has been gained in its use, so that a realistic assessment of required mesh sizes for specific requirements can be made.

It might be noted that finite difference approximations can be developed for the equations of continuity and motion, Eqs. 3.2, 3.3, quite independently of the specific form of the constitutive relation. A computer code could thus be developed which contains the kinematical relations, the logic necessary to determine gradients, and to control the progress of the calculation through the space-time net, without reference to a specific constitutive relation, leaving provision to insert a specific constitutive relation as a sub-program. One requirement is that the constitutive relation must provide a unique stress increment corresponding to a specified strain increment at each step of the calculation, and that these increments be of the same sign. Thermodynamic and nonmechanical processes can be included in the sub-program, as can provision of a material "memory" for the control of strain hardening and similar effects. This is a considerable convenience. The latter processes are open to some speculation, while the kinematical relations are, of course, well established. The constitutive relation could thus be altered very simply by replacing a sub-program, without affecting the main program.

4. CONSTITUTIVE RELATION

The form of the constitutive relation Eq. 3.4 is not completely arbitrary. As is well known, certain integrability conditions must be satisfied. Invariance of the constitutive relation under rigid body motion demands that only the symmetric part of the deformation gradient $u_{(i,j)}$ may enter, since the antisymmetric part $u_{[i,j]}$ represents a rigid body rotation, for which the reaction of the material should not alter. Furthermore, invariance under coordinate transformation requires that stress and deformation tensors should enter only through their scalar invariants.

Except under very special circumstances, the state of the material is not uniquely described by kinematic quantities alone. It is necessary to introduce thermodynamic quantities and to consider the interaction between the kinematics and thermodynamics. While numerous studies have been reported, the thermodynamic treatment of a solid is still in a somewhat speculative stage.

To provide a basis for discussion, we present the salient points of the thermodynamic argument. Following Truesdell,¹² the internal energy is introduced as a set function to balance the total energy, entropy is introduced as a dimensionally independent state variable which, together with the substate, suffices to determine the internal energy, thus leading to the concept of a caloric equation of state. Temperature and thermodynamic tensions are introduced as derived variables, and equations for entropy production follow. The material description is completed by supplying phenomenological relations which govern dissipative processes.

We do not pursue arguments of the existence of a caloric equation of state, or of the applicability of the equations for entropy production for nonequilibrium processes. These are discussed at length in most thermodynamic texts.^{13,14}

4.1 Thermodynamic Relations

In addition to the fundamental principles of conservation of mass and momentum, which lead to the equations of continuity and motion, we have the principle of conservation of energy. We may write for a body

$$\dot{K} + \dot{E} = W + D \quad (4.1)$$

where K is the kinetic energy of the body, W the mechanical power applied to its surface, and D the non-mechanical power supplied to the body. The quantity E is an additive set function, termed the internal energy, such that the total energy ($K + E$) is balanced. Evaluating each of these terms in integral form, the result, when simplified by the use of the equation of motion, is for the element dx_i

$$\rho \dot{\mathcal{E}} = \tau_{ij} \dot{u}_{i,j} - h_{k,k} + \rho q \quad (4.2)$$

The increase in internal energy is thus due to the external supply of non-mechanical energy Q_E , where

$$Q_E = \rho q - h_{k,k} \quad (4.3)$$

and the external stress work, P_E given by

$$P_E = \tau_{ij} \dot{u}_{i,j} \quad (4.4)$$

Here \mathcal{E} is the specific internal energy $E = \int_V \mathcal{E} dm$, h denotes the efflux of nonmechanical energy, and q is the supply of nonmechanical energy (by arbitrary sources and sinks).

It is sometimes convenient to decompose the stress and deformation rate tensors into spherical and deviatoric components. By virtue of Eq. 3.2 we obtain for P_E

$$P_E = p \dot{\rho}/\rho + \tau'_{ij} \dot{u}'_{(i,j)} \quad (4.4a)$$

where the prime denotes the deviator, and the spherical stress $1/3 \tau_{kk} = -p$, where p is the pressure.

There are a number of parameters which affect the internal energy. Some of these are the deformation gradients $u_{i,j}$. In general, let there be M parameters influencing the internal energy, denoted by v_α . The set \underline{v} is regarded as given a priori, and defines the thermodynamic substate.

The basic assumption of thermodynamics is that the substate plus a single other dimensionally independent scalar parameter suffice to determine \mathcal{E} without reference to time, place, motion, or stress, i.e. we postulate the existence of a caloric equation of state

$$\mathcal{E} = f(s, \underline{v}) \quad (4.5)$$

where s is the specific entropy.

The temperature T and thermodynamic tensions τ_α are defined as

$$T \equiv \frac{\partial \mathcal{E}}{\partial s} \quad (\underline{v} \text{ constant}) \quad (4.6)$$

$$\tau^\alpha \equiv \frac{\partial \mathcal{E}}{\partial v_\alpha} \quad (s, v_1 \dots v_{\alpha-1}, v_{\alpha+1} \dots v_M \text{ constant})$$

In view of Eq. 4.5

$$T = T(s, \underline{v}) \quad \tau^\alpha = \tau^\alpha(s, \underline{v}) \quad (4.7)$$

If all functions so far introduced are regular and reversible, then as consequences of Eq. 4.5, 4.7

$$s = s(T, \underline{v}) \quad \varepsilon = \varepsilon(T, \underline{v}) \quad \tau^\alpha = \tau^\alpha(T, \underline{v}) \quad (4.8)$$

These equations are known as thermal equations of state, any one of which is insufficient to determine all the state functions.

In our case, the definitions Eq. 4.6 imply

$$\rho \dot{\varepsilon} = \rho \tau^\alpha \dot{v}_\alpha + \rho T \dot{s} \quad (4.9)$$

This equation, in parallel with Eq. 4.2 implies that the increase in internal energy is due to the inner stress power P_I and the inner supply of non-mechanical energy Q_I , i.e.

$$\rho \dot{\varepsilon} = P_I + Q_I \quad (4.10)$$

where

$$P_I = \rho \tau^\alpha \dot{v}_\alpha \quad (4.11)$$

and

$$Q_I = \rho T \dot{s} \quad (4.12)$$

From Eqs. 4.2, 4.12 we have an expression for the production of entropy

$$\rho T \dot{s} = P_E - P_I + Q_E \quad (4.13)$$

Thus the production of entropy is due not only to the external heat energy flux, but also to the excess of external power over the internal power, i.e., to the dissipative work done. We express this latter quantity in the general form

$$P_E - P_I = \rho \gamma^\alpha \dot{v}_\alpha \quad (4.14)$$

Equation 4.13 may now be expressed in the form

$$\rho T \dot{s} = \rho \gamma^\alpha \dot{v}_\alpha - h_{k,k} + \rho q \quad (4.15)$$

or rearranging

$$\rho \dot{s} + \left(\frac{h_k}{T} \right)_{,k} - \frac{\rho q}{T} = \frac{\rho}{T} \gamma^\alpha \dot{v}_\alpha - \frac{h_{k,T,k}}{T^2} \quad (4.16)$$

The terms on the left represent the "reversible" rate of entropy flow into the element, while the terms on the right are usually considered to be the "irreversible" part of the entropy production. This may be written

$$\Gamma = \frac{\rho}{T} \gamma^\alpha \dot{v}_\alpha - h_k \left(\frac{1}{T} \right)_{,k} \quad (4.17)$$

and is subject, by the second law of thermodynamics, to the inequality

$$\Gamma \geq 0 \quad (4.18)$$

The results are indefinite until specific quantities are inserted for the substate \underline{v} . We follow usual practice in irreversible thermodynamics and generalize still further. Noting that the "irreversible" entropy production may be regarded as the sum of products of generalized thermodynamic forces $F^\alpha, (\dot{v}_\alpha, (1/T)_{,k})$ and corresponding generalized thermodynamic fluxes $f_\alpha, (\rho \gamma^\alpha / T, -h_k)$, Eq. 4.17 takes the bilinear form

$$\Gamma = F^\alpha f_\alpha \quad (4.19)$$

which by Eq. 4.18 is positive definite (or zero).

It is necessary to express the relationships between the fluxes and forces. The assumption is usually made¹⁴ that the equations for entropy production are only valid for states very near thermodynamic equilibrium ($\Gamma = 0$) for which the relations between the forces and fluxes may be regarded as linear, whatever their actual form, i.e. there exist phenomenological relations of the form

$$F^\alpha = L^{\alpha\beta} f_\beta \quad (4.20)$$

The Onsager reciprocal relations, following from the property of microscopic time reversal invariance, state that the matrix of coefficients $L^{\alpha\beta}$ is symmetric. Thus

$$\frac{\partial F^\alpha}{\partial f^\beta} = \frac{\partial F^\beta}{\partial f^\alpha} \quad (4.21)$$

i.e., the forces are "irrotational" in the flux space, which is the necessary and sufficient condition for the existence of a dissipation potential Φ such that

$$F^\alpha = \frac{\partial \Phi}{\partial f^\alpha} \quad (4.22)$$

The potential has the form

$$\Phi = \frac{1}{2} L^{\alpha\beta} f_\alpha f_\beta = \frac{1}{2} F^\alpha f_\alpha = \frac{1}{2} \Gamma \quad (4.23)$$

The dissipation potential is therefore one half of the irreversible entropy production.

While a general extension of Onsager's relations to nonlinear phenomenological relations has not been forthcoming, Ziegler¹⁵ on the basis of certain orthogonality conditions, postulates the existence of a potential Eq. 4.22 for general relations of the form

$$F^{\alpha} = F^{\alpha}(\underline{f}) \quad (4.24)$$

We now discuss two important constraints. Consider first a process in which no dissipative work is being done, i.e. $P_E = P_I$. A necessary and sufficient condition is, from Eq. 4.13

$$\rho T \dot{s} = \rho q - h_{k,k} \quad (4.25)$$

A sufficient condition is $\rho q - h_{k,k} = 0$, i.e. adiabatic conditions, which in the present case also implies $\dot{s} = 0$, i.e. isentropic conditions. Equation 4.2 reduces to

$$\rho \dot{\mathcal{E}} = \tau_{ij} \dot{u}_{i,j} \quad (4.26)$$

and the internal energy is thus a nondissipative strain energy potential. In view of Eqs. 4.8 it is thus possible to eliminate thermodynamic quantities from explicit consideration, and to characterize the material by a stress-strain relation of the form

$$f(\tau_{ij}, u_{(i,j)}) = 0 \quad (4.27)$$

for a specific value of entropy.

We introduce the free energy, defined by

$$\psi \equiv \mathcal{E} - sT \quad (4.28)$$

From Eq. 4.9 we have

$$\rho \dot{\psi} = \rho \tau^{\alpha} \dot{v}_{\alpha} - \rho s \dot{T} \quad (4.29)$$

Thus

$$s = - \frac{\partial \psi}{\partial T} \quad (\underline{v} \text{ constant}) \quad (4.30)$$

$$\tau^\alpha = \frac{\partial \psi}{\partial v_\alpha} \quad (T, v_1 \cdot \cdot \cdot v_{\alpha-1}, v_{\alpha+1} \cdot \cdot \cdot v_M \text{ constant})$$

and in view of Eq. 4.8

$$\psi = \psi(T, \underline{v}) \quad (4.31)$$

For nondissipative process, i.e., $P_E = P_I$, and the constraint $\dot{T} = 0$, i.e., isothermal conditions

$$\rho \dot{\psi} = \tau_{ij} \dot{u}_{i,j} \quad (4.32)$$

and the free energy is a strain energy potential. In view of Eq. 4.31 it is again possible to eliminate thermodynamic quantities from explicit consideration and to characterize the material by a stress-strain relation of the type Eq. 4.27, but for a specific value of temperature.

4.2 Special Cases

The equations in the previous subsection are quite general, but also indefinite, until specific quantities v_α are inserted. In this section we mention some of the specialized theories of interest in the present context and their connection to the general theory.

Consider a body deformed from some initial state by suitable surface tractions. If the body is subsequently unloaded and returned to its initial temperature, generally only part of the deformation will disappear. Observation of the body in the large does not give an indication of the behavior of the material, however. It is necessary to isolate a volume element from the body by cutting, to

remove the influence of residual stresses imposed by the surrounding material, if we wish to observe the material in its natural state.¹⁶ Observing such an element, we assume that after the application of a stress, if the element is returned to its initial state of stress and temperature, then only a part of the deformation will disappear. We term this part the recoverable deformation. The remaining deformation will be termed the irrecoverable deformation. The recoverable part of the deformation will be limited by a yield condition.

For a motion in which all deformations are recoverable, we assume that the substate \underline{v} is defined by the six deformation gradients $u_{(i,j)}$. We divide the total stress into two components

$$\tau_{ij} = \tau_{ij}^e + \tau_{ij}^D \quad (4.33)$$

where τ_{ij}^e is the nondissipative stress, so that

$$P_I = \tau_{ij}^e \dot{u}_{i,j} \quad (4.34)$$

and the remaining stress τ_{ij}^D is called the dissipative stress. Then the equation for entropy production, Eq. 4.15 becomes

$$\rho T \dot{s} = \tau_{ij}^D \dot{u}_{i,j} - h_{k,k} + \rho q \quad (4.35)$$

Solution of motions in general elastic media with viscous dissipation and heat conduction then may proceed from the equation of continuity Eq. 3.2, the equation of motion Eq. 3.3, the energy equation Eq. 4.2, and the entropy equation, Eq. 4.35, together with a caloric equation of state Eq. 4.5 (or two thermal equations of state Eqs. 4.8₁, 4.8₂) and phenomenological relations connecting τ_{ij}^D and $\dot{u}_{(i,j)}$ and h_k and $(1/T)_{,k}$ subject to specific initial and

boundary conditions. Body forces F_j and heat sources q are considered to be specified a priori.

We now consider motions which include irrecoverable deformation. In addition to the total deformation gradients, the substate \underline{y} must include functions of the irrecoverable part of the deformation. While this case is receiving considerable attention,^{15, 17, 18, 19} a satisfactory general treatment has not yet been given. We therefore consider some special assumptions which allow simplification.

In the "hydrodynamic" theory,^{*} it is assumed that the motion is adiabatic ($Q_E = 0$) and that the deviatoric stress work is negligible compared to the spherical stress work. With these assumptions, Eq. 4.2 using the decomposition Eq. 4.4a becomes

$$\rho \dot{E} = p \dot{p}/p \quad (4.36)$$

In most compacted materials, a purely spherical deformation is almost all recoverable. Neglecting volume viscosity, Eq. 4.15 reduces to the statement $\dot{s} = 0$, and the motion is isentropic. These are the equations for a perfect fluid.

At possible shock waves, dissipative mechanisms come into play so that entropy changes may occur. These can be handled by applying the Rankine-Hugoniot relations⁴ to calculate the entropy jump at a shock. When the strength of a shock wave varies as it propagates, the motion between

*The term hydrodynamic is unfortunate since the theory depends on the compressibility of the medium for its basic assumption, while the term hydrodynamic has hitherto been applied specifically to incompressible fluid flow. Since the term is in fairly wide use in the present context, it will be retained here.

shock waves is isentropic but not homoentropic ($s_{,i} \neq 0$):

Frequently an artificial viscous dissipation term is added to Eq. 4.36 which is so formulated that it is negligibly small except in regions of very high gradients, i.e. at shock waves. This well-known mathematical device renders the solution continuous and makes the equations amenable to solution by finite difference numerical integration methods, while preserving correct isentropic flow away from shock waves.⁸ Equation 4.36 takes the form

$$\rho \dot{\epsilon} = p \dot{\rho}/\rho + \tau'_{ij} \dot{u}'_{(i,j)} \quad (4.37)$$

Noting Eq. 4.8 and the fact that the only quantity entering the substate \underline{v} is ρ , so that $\underline{\tau}$ reduces to a function of p , we have the thermal equation of state

$$\epsilon = \epsilon(p, \rho) \quad (4.38)$$

We may proceed from Eqs. 3.2, 3.3, and 4.38 without explicit reference to the entropy equation.

Temperatures may be estimated if, in addition, the specific heat at constant pressure ($p = 0$) is known.¹

It is clear that the hydrodynamic theory is only a reasonable approximation at very high pressures when the yield stress is negligible by comparison. A further condition is that the deviatoric deformation rate be of the same order as the spherical deformation rate so that the deviatoric stress work is in fact negligible compared to the spherical stress work.

We turn now to the problem of the mathematical description of the transient response of a solid body to intense impulsive loads.

Part of the motion will occur under very high pres-

sure. Provided that the yield condition is not altered so that the material is never able to support a very large deviatoric stress, the hydrodynamic theory will provide a reasonable approximation to this phase of the motion.

Part of the transient motion will, however, occur at relatively low pressure, where the hydrodynamic assumption will be violated. If the low pressure part of the motion contributes significantly to the overall response, then the hydrodynamic theory will be inadequate to describe the motion.

It will be necessary, under such conditions, to consider a full thermodynamic treatment of the problem. Such a treatment has so far not been given.

In the interim, it may be useful to attempt to incorporate material strength as an approximate correction to the hydrodynamic theory, without explicit reference to the entropy balance equation. While this might be done in a number of different ways, one might make the following assumptions:

- a) The motion is adiabatic.
- b) The yield stress is not increased by a very large amount at any time in the motion. During any high pressure phase of the motion, the treatment will then be equivalent to that of the hydrodynamic theory, since the yield stress will be negligible compared to the pressure.
- c) The energy dissipation in irrecoverable (plastic) deformation is negligible compared to the energy stored in recoverable (elastic) deformation. The motion will thus, in effect, be assumed to be isentropic. This is justified (providing assumption b is valid) in the high pressure phase of the motion, but may be seriously in error if a large amount of irrecoverable deformation occurs during the low pressure phase of the motion. Entropy changes may

still occur on shock waves where they are handled by the Rankine-Hugoniot relations, as in the hydrodynamic theory.

d) The material properties can hardly be assumed to be temperature independent. However, we assume that the temperature may be estimated from the spherical deformation as in the hydrodynamic theory. This entails two assumptions, viz., that the energy stored in deviatoric recoverable deformation is negligible, which follows from assumption b above, and that the temperature increase due to energy dissipation in irrecoverable deformation is negligible, which follows from assumption c above.

Denoting by \underline{u}^e the recoverable part of the deformation, the energy equation, Eq. 4.2, thus becomes

$$\rho \dot{\mathcal{E}} = \tau_{ij} \dot{u}_{i,j}^e \quad (4.39)$$

while the entropy equation again reduces to $\dot{s} = 0$ between shocks. Equation 4.39 expresses the fact that there is an elastic strain energy potential $\rho \mathcal{E} = \Psi$ such that

$$\tau_{ij} = \frac{\partial \Psi}{\partial u_{i,j}^e} \quad (4.40)$$

and the constitutive relation reduces to a stress-strain relation involving the recoverable part of the deformation. Also needed is a yield condition to determine, in effect, the proportion of the deformation which is recoverable.

The elastic stress-strain relation is applicable to a specific value of entropy. Provision must be made for entropy production at shock waves with a corresponding change in the strain energy potential.

5. ELASTIC BEHAVIOR

5.1 Finite-Strain Theory

In this section we investigate the possibility of describing the recoverable part of the deformation in terms of finite-strain elasticity theory.²⁰ Following usual practice, we employ the Eulerian strain tensor

$$\epsilon_{kj} = \frac{1}{2}(u_{k,j} + u_{j,k} - u_{i,j} u_{i,k}) \quad (5.1)$$

Between shocks, the motion is isentropic ($\dot{s} = 0$) and there exists a strain-energy potential $\Psi = \rho \mathcal{E}$. By Eqs. 4.5 and 5.1, the strain-energy potential is a function of strain only, at constant entropy $\Psi = \Psi(\underline{\epsilon})$. This function is represented, for an isotropic homogeneous medium, by a power series expansion in the strain invariants.

$$\Psi = aI_1 + bI_1^2 + cI_2 + \ell I_1^3 + mI_1 I_2 + nI_3 + \dots \quad (5.2)$$

where

$$\begin{aligned} I_1 &= \epsilon_{ii} & I_2 &= \frac{1}{2}(\epsilon_{ij} \epsilon_{ji} - \epsilon_{ii} \epsilon_{jj}) \\ I_3 &= \frac{1}{6}(2\epsilon_{ij} \epsilon_{jk} \epsilon_{ki} - 3\epsilon_{ij} \epsilon_{ji} \epsilon_{kk} + \epsilon_{ii} \epsilon_{jj} \epsilon_{kk}) \end{aligned}$$

To conform with the requirement that the stress vanish in the unstrained state and that the theory agrees with the classical theory for infinitesimal strain, we set $a = 0$, $b = (\lambda + 2\mu)/2$, $c = -2\mu$ where λ , μ are the adiabatic Lamé constants.

With the definition Eq. 5.1, the stress-strain law may be obtained from the strain energy potential by

$$\tau_{ij} = (\delta_{ik} - 2\epsilon_{ik}) \left(\frac{\partial \Psi}{\partial \epsilon_{kj}} \right)_s \quad (5.3)$$

Including only terms shown in Eq. 5.2, the stress-strain law thus takes the form

$$\begin{aligned} \tau_{rs} = \rho/\rho_0 \{ & \lambda I_1 \delta_{rs} + 2\mu \epsilon_{rs} - 4\mu \epsilon_{ri} \epsilon_{is} \\ & - (2\lambda + m) I_1 \epsilon_{rs} + (3\ell + m) I_1^2 \delta_{rs} + m I_2 \delta_{rs} \\ & + n \zeta_{rs} I_3 - 2(3\ell + m) I_1^2 \epsilon_{rs} + 2m \epsilon_{ri} \epsilon_{is} I_1 \\ & - 2m I_2 \epsilon_{rs} - 2n \delta_{rs} I_3 \} \end{aligned} \quad (5.4)$$

where

$$\zeta_{ri} \epsilon_{is} = \delta_{rs}$$

and the density ratio is given by

$$\rho/\rho_0 = (1 - 2I_1 + 4I_2 - 8I_3)^{1/2} \quad (5.5)$$

If the material constants λ , μ , ℓ , m , n can be found, Eq. 5.4 and Eq. 5.5 provide a complete description of the elastic behavior at constant entropy to within the approximation implied by Eq. 5.2.

If the value of entropy is changed, the numerical values of the coefficients will be altered.

Note that a parallel development holds for isothermal conditions ($\dot{T} = 0$) when, from Eq. 4.32, there exists a strain energy potential $\Psi = \rho \psi(\underline{\epsilon})$. This strain energy potential may be expanded in the same way as in Eq. 5.2, leading to an identical stress-strain relation Eqs. 5.4, 5.5

except that the constants λ , μ , ℓ , m , n now apply to isothermal conditions. Their numerical values will be different from those of the isentropic theory and will be altered if the value of temperature is altered.

To complete the description of the material, the coefficients λ , μ , ℓ , m , n must be evaluated. Before proceeding we write down some further results which will be useful.

For a purely spherical strain $\epsilon_{ii}/3 = e$, the relation between spherical strain and spherical stress or pressure $\tau_{ii}/3 = -p$ can easily be found from Eq. 5.4

$$-p = e(1 - 2e)^{5/2} \left\{ (3\lambda + 2\mu) + e(27\ell + 9m + n) \right\} \quad (5.6)$$

while the density ratio Eq. 5.5 becomes

$$\rho/\rho_0 = (1 - 2e)^{3/2} \quad (5.7)$$

Combining Eq. 5.6 and 5.7 gives

$$\begin{aligned} p = \frac{1}{2} \left\{ \left(\frac{\rho}{\rho_0} \right)^{7/3} - \left(\frac{\rho}{\rho_0} \right)^{5/3} \right\} & \left\{ (3\lambda + 2\mu) + \frac{1}{2}(27\ell + 9m + n) \right. \\ & \left. - \frac{1}{2}(27\ell + 9m + n) \left(\frac{\rho}{\rho_0} \right)^{2/3} \right\} \end{aligned} \quad (5.8)$$

It is desirable to derive the elastic moduli for a material subjected to a spherical strain. We follow the method of Birch.²¹

If a small perturbation is produced in a state of uniform spherical strain, then we may write

$$\epsilon_{rs} = e \delta_{rs} + e_{rs} \quad (5.9)$$

where e_{rs} , the perturbation strain, is assumed to be infinitesimal. The strain invariants of e_{rs} may then be written

$$\begin{aligned} I_1 &= 3e + I_1' \\ I_2 &= 3e^2 + 2eI_1' + I_2' \\ I_3 &= e^3 + e^2 I_1' + eI_2' + I_3' \end{aligned}$$

where the invariants I_1' , I_2' , I_3' refer to e_{rs} . Neglecting the second and third invariants of e_{rs} and inserting in Eq. 5.5 we find, approximately

$$\rho/\rho_0 = (1 - 2e)^{3/2} \left\{ 1 - \frac{I_1'}{1 - 2e} \right\} \quad (5.10)$$

Decomposing the stress in a similar way to the strain we write

$$\tau_{rs} = -p \delta_{rs} + \sigma_{rs} \quad (5.11)$$

where the perturbation stress σ_{rs} corresponds to the strain e_{rs} , and we find a relation between σ_{rs} and e_{rs} analogous to Eq. 5.4

$$\begin{aligned} \sigma_{rs} &= (1 - 2e)^{3/2} \left\{ \lambda - e(5\lambda + 2\mu - 18l - 7m - n) \right. \\ &\quad \left. - e^2 (63l + 23m + 3n) \right\} \delta_{rs} I_1' \\ &\quad + (1 - 2e)^{3/2} \left\{ 2\mu - e(6\lambda + 3\mu + 3m + n) \right. \\ &\quad \left. - e^2 (54l + 12m) \right\} e_{rs} \end{aligned} \quad (5.12)$$

Denoting the total displacement of a point P

as $\underline{u} = \underline{u}_0 + \underline{u}'$, where \underline{u}_0 is due to the spherical strain, and \underline{u}' is due to the perturbation, we find

$$u_{i,j} = \delta_{ij} \left\{ 1 - (1 - 2e)^{1/2} \right\} + (1 - 2e)^{1/2} u'_{i,j}$$

Inserting this in Eq. 5.1 and neglecting powers of \underline{u}' higher than the first, we obtain

$$e_{ij} = u'_{(i,j)} (1 - 2e) \quad (5.13)$$

To obtain the shear modulus, we set $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = \sigma_{31} = 0$ and $u'_2 = u'_3 = 0$. Then

$$G = \frac{\sigma_{12}}{u'_{1,2}}$$

From Eqs. 5.12, 5.13 this becomes

$$G = \frac{1}{2}(1 - 2e)^{5/2} \left\{ 2\mu - e(6\lambda + 8\mu + 3m + n) - e^2(54\ell + 12m) \right\} \quad (5.14)$$

To obtain the bulk modulus we set $\sigma_{12} = \sigma_{23} = \sigma_{31} = 0$ and $\sigma_{11} = \sigma_{22} = \sigma_{33}$, $u'_1 = u'_2 = u'_3$. Then

$$K = \frac{\sigma_{11}}{u'_{1,1}}$$

which from Eqs. 5.12, 5.13 becomes

$$K = \frac{1}{3}(1 - 2e)^{5/2} \left\{ (3\lambda + 2\mu) - e(21\lambda + 14\mu - 54\ell - 18m - 2n) - e^2(243\ell + 81m + 9n) \right\} \quad (5.15)$$

To obtain the uniaxial strain modulus, we set $\sigma_{22} = \sigma_{33}$ and $u_2 = u_3 = 0$. Then

$$F = \frac{\sigma_{11}}{u'_{1,1}}$$

which from Eqs. 5.12, 5.13 becomes

$$F = (1 - 2e)^{5/2} \left\{ (\lambda + 2\mu) - e(11\lambda + 10\mu - 18\ell - 4m) - e^2(117\ell + 35m + 3n) \right\}^{1/2} \quad (5.16)$$

We may note that $F = K + 4G/3$, i.e. the moduli bear the same relationship to one another as in the infinitesimal classical theory.

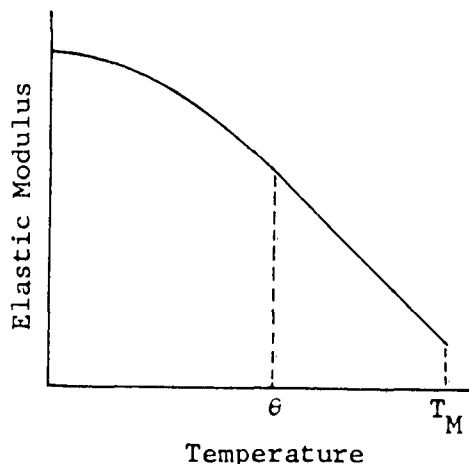
The corresponding infinitesimal shear, bulk and longitudinal wave velocities are $\sqrt{G/\rho}$, $\sqrt{K/\rho}$ and $\sqrt{F/\rho}$.

A more elegant derivation has recently been given by Toupin and Bernstein²² and Hayes and Rivlin²³ from the consideration of jump conditions at a finite amplitude wave.

5.2 Evaluation of Elastic Constants

Most of the recent experimental determinations of adiabatic elastic moduli have been made using the ultrasonic pulse-echo technique.^{24,25} Measurements have been made for a large variety of materials, both polycrystalline and in single crystal form, at ambient temperature and at zero pressure.

A number of studies have included a measurement of the dependence of the elastic moduli on temperature. It has been shown that, for many materials, the elastic moduli show similar behavior. (See sketch)



Above the Debye temperature θ the elastic moduli generally show an almost linear decrease with temperature to a finite value at the melting temperature T_M . Below the Debye temperature, the slope of the curve decreases, approaching zero at zero temperature.

Measurements of elastic moduli at ambient pressure (1 atm) lead to values of the first order elastic constants. In the case of an isotropic medium, these are the Lamé constants.

Several studies have been performed under hydrostatic pressure and the dependence of the elastic moduli on pressure has been reported.^{26,27,28,29} Due to limitations in the technique, pressures have been limited to about 10 kb. Nevertheless, the data are sufficient to determine the pressure derivatives of the elastic moduli. Most of these determinations have been for single crystal specimens.

In an isotropic medium, only two moduli can be determined; the longitudinal modulus F corresponding to the longitudinal wave speed, and the shear modulus G corresponding to the transverse wave speed. The measured pressure derivatives would then give two relations for the second order elastic constants l , m , and n through Eqs. 5.14 and 5.15. These relations are insufficient to determine l , m , and n separately.

Hughes and Kelley³⁰ in addition measured wave velocities under conditions of uniaxial stress and thus obtained

sufficient information to determine ℓ , m , and n separately, in a manner somewhat analogous to that subsequently suggested by Toupin and Bernstein.²² However, the serious disadvantage of this method is that the static uniaxial stress cannot be carried to high values without causing permanent deformation in most materials. The values of the second-order constants determined by Hughes and Kelley for polystyrene, Armco iron, and pyrex are correspondingly very uncertain.

Information relevant to the second-order elastic constants may also be deduced from adiabatic compressibility data under spherical stress, through Eq. 5.8, or equivalently through Eq. 5.15, noting that $K = \rho(\partial P/\partial \rho)_S$. Hydrostatic compression experiments have been carried out to pressures of 100 kb, but refer to isothermal conditions. Under the assumption that the yield stress remains small, so that the deviatoric stress may be neglected in comparison with the spherical stress at high pressure, the shock wave and material particle velocity measurements of McQueen and Marsh³¹ and of Altshuler et al³² in plate impact experiments may be reduced to pressure-density information through use of the Rankine-Hugoniot relations. Such experiments have been carried out to pressures of about 4 Mb, but refer to the shock Hugoniot. Adiabatic compressibility data may be approximately inferred from both the isothermal data and hugoniot data by use of the Mie-Grüneisen equation.¹ Compressibility data on the zero temperature isotherm may be similarly inferred.

If the neglect of terms of order greater than ϵ^3 (Eq. 5.2) is justified, then we see from Eq. 5.8 that a plot of

$$\frac{P}{\left(\frac{\rho}{\rho_0}\right)^{5/3} - \left(\frac{\rho}{\rho_0}\right)^{7/3}}$$

vs. $(\rho/\rho_0)^{2/3}$ should be a straight line. Moreover the values of $(\lambda + 2\mu/3)$ and $(27\ell + 9m + n)$ may be found from the slope of the line and its intercept with the ordinate.

We have, so far, made only a preliminary comparison for copper. Data for the zero temperature isotherm up to 2.7 Mb are available, and are shown plotted in figure 5.1. The high pressure points are those of Altshuler,³² while the low pressure points were taken from a recent reintegration of the data of McQueen and Marsh.⁶ It is very encouraging that the data can be fitted adequately by a straight line, suggesting that additional terms are not required in the expansion, Eq. 5.2. The straight line fit yields $(\lambda + 2\mu/3) = 1440$ kb, and $(27\ell + 9m + n) = -5880$ kb.

Data for adiabatic conditions are available only up to 500 kb; (Walsh et al¹). Two adiabats are plotted in figure 5.2, one passing through ambient conditions ($P = 0$, $T = 293$ °K) and one passing through a point at 500 kb on the Hugoniot, and thus referring to a different value of (constant) entropy. The straight line fits yield $(\lambda + 2\mu/3) = 1403$ kb, $(27\ell + 9m + n) = -5840$ kb and $(\lambda + 2\mu/3) = 1387$ kb $(27\ell + 9m + n) = -4460$ kb, respectively. The value of $(\lambda + 2\mu/3) = 1403$ kb on the adiabat through ambient conditions may be compared to a value of $(C_{11} + 2C_{12})/3 = 1370$ kb measured by Daniels and Smith²⁷ by the pulse-echo technique, and a value of $K_s = 1414$ kb found from static compressibility data corrected to adiabatic conditions.³³

Note that, so far, the comparisons in figures 5.1 and 5.2 constitute only a new empirical fit to the compressi-

bility data, the only interesting fact to emerge is that it is apparently sufficient to include terms only up to those shown in Eq. 5.2 to fit the data up to 4 Mb. To demonstrate the existence of a strain energy potential, and the applicability of the foregoing theory, it would be necessary to measure the velocities of all nine possible kinds of waves in a medium subjected to an arbitrary homogeneous deformation, and verify that they satisfy the relevant compatibility conditions.³⁴ If the yield stress is negligibly small at high pressure, however, only two distinct wave velocities are observable.

Altshuler et al.,³⁵ have reported an experiment in which both these wave velocities have been measured. The "overtaking-relaxation method" yielded the speed of the bulk wave, which corresponded to $\sqrt{K/\rho}$, determined from the adiabatic compressibility data. The "lateral relaxation method" led to velocities which were higher than the bulk wave velocity and corresponded to the longitudinal wave velocity. Only one measurement is available for copper. The measured velocity of a longitudinal release wave propagating into material shock loaded to 407 kb was found to be 6.33 Km/sec, yielding $K + 4G/3 = F = 4290$ kb. We have values of $(\lambda + 2\mu/3)$ and $(27\ell + 9m + n)$ for isentropes through 500 kb and 0 kb on the hugoniot only, but by rough interpolation the isentropic bulk modulus at 407 kb is found to be $K = 3160$ kb. Thus $G = 850$ kb approximately, which may be compared to the value of $\mu = 478$ kb at zero pressure. Although both the measurement and the comparison are very rough, there seems to be a substantial increase in shear modulus with pressure. Careful measurement of the velocities of longitudinal release waves propagating into material shock compressed to a variety of pressures should provide sufficient information to determine

if Eq. 5.14 is adequate and to provide an independent determination of another combination of the second-order elastic constants.

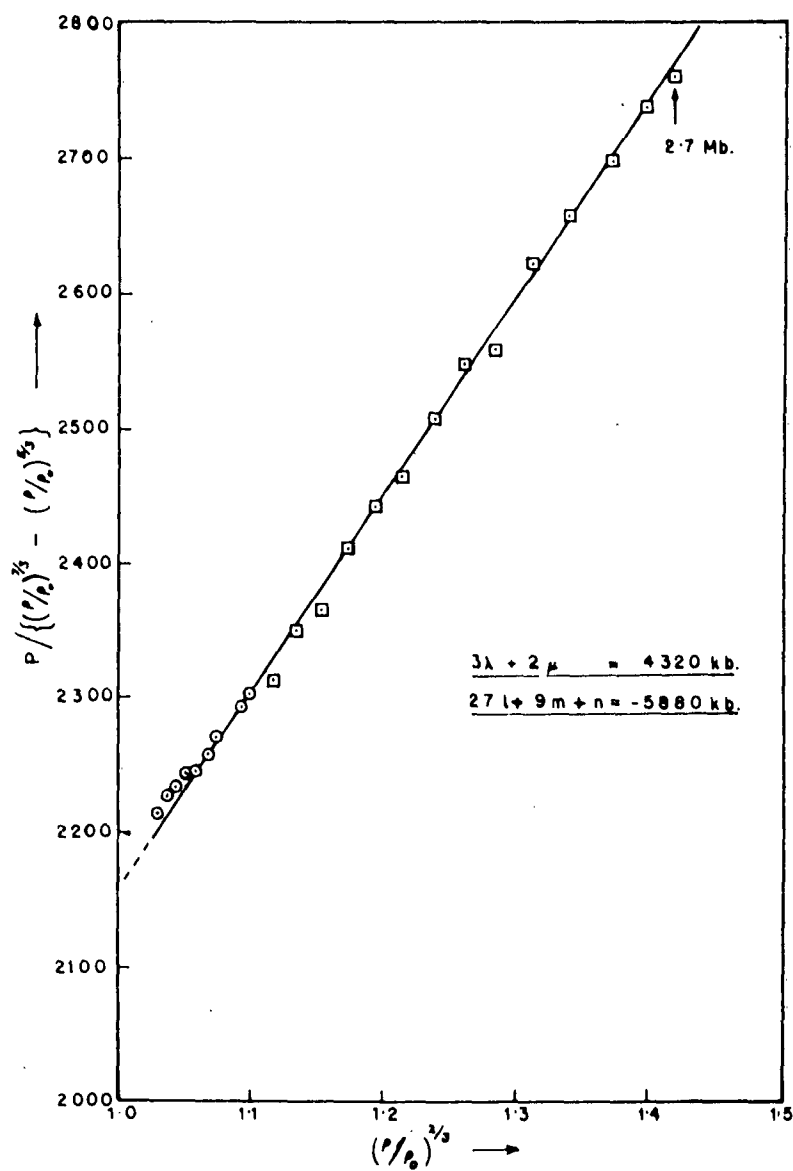


Figure 5.1. Compressibility data for the zero temperature isotherm for copper to 2.7 Mb.

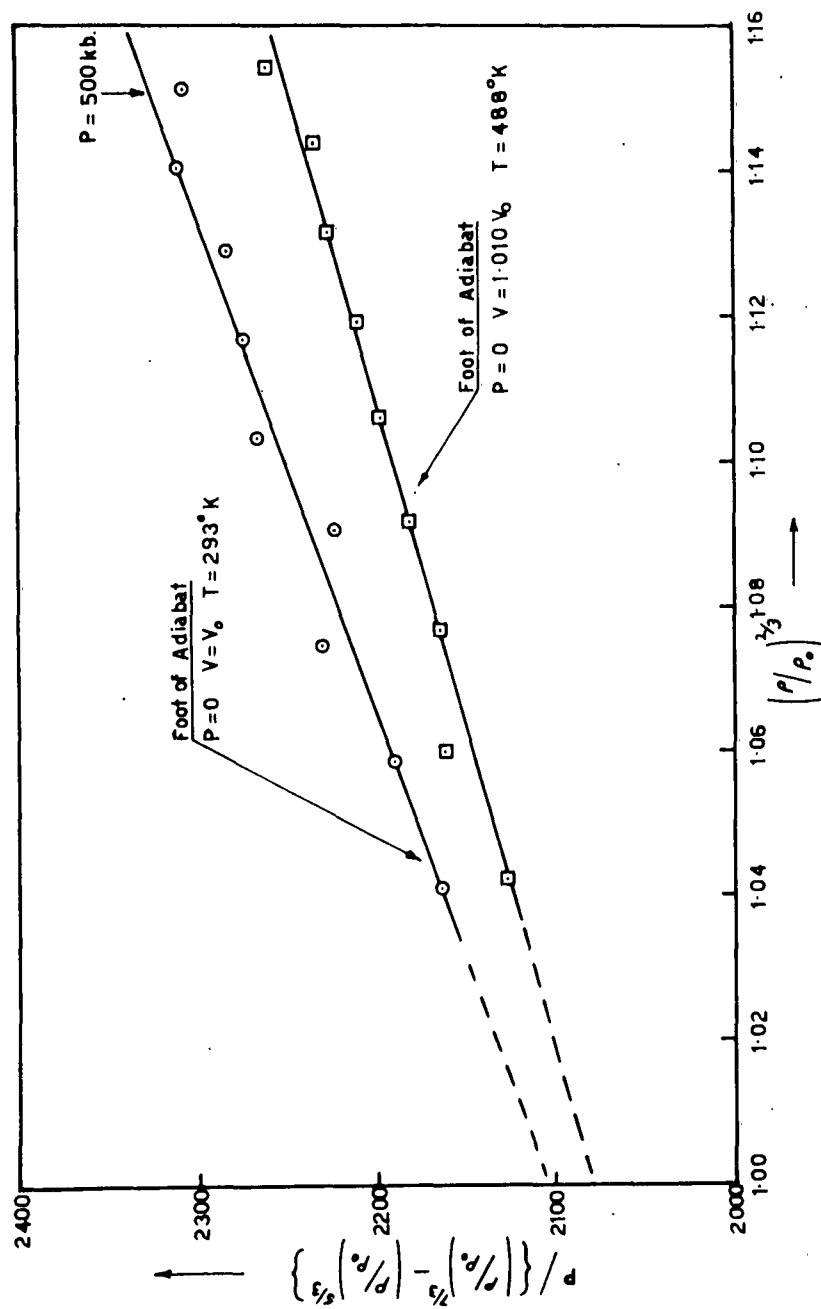


Figure 5.2. Compressibility data for two isentropes for copper to 500 kb.

6. YIELD BEHAVIOR

6.1 The Mechanics of Yield Behavior

In this section we investigate certain aspects of the yield condition relevant to the problem at hand. To avoid unnecessary complication due to finite strain effects, the initial development is restricted to infinitesimal strain, and comparisons will be made with experiments in which the compression is very small. The development is also limited to isotropic, homogeneous media.

We recall the stress decomposition

$$\tau_{ij} = \tau \delta_{ij} + \tau'_{ij} \quad (6.1)$$

The spherical stress $\tau = 1/3 \tau_{ii} = -p$ is here taken to be nondissipative. The stress deviator is further divided into two components.

$$\tau'_{ij} = \sigma'_{ij} + \tau^*_{ij} \quad (6.2)$$

where τ^* is a (viscous) overstress, which depends on the strain rate and is zero when the strain rate is zero.

The strain is divided into two parts, a recoverable elastic strain ϵ^e and an irrecoverable plastic strain ϵ^p , i.e.

$$\epsilon_{ij} = \epsilon^e_{ij} + \epsilon^p_{ij} \quad (6.3)$$

In conformity with the stress decomposition, we assume that the spherical strain is entirely elastic and that the plastic strain involves no volume change and is thus independent of pressure.

It is consistent with the concept of an equilibrium yield stress and a transient overstress to describe the stress-strain behavior in the manner suggested by Malvern.³⁶ We can generalize the uniaxial description to a triaxial description as follows:

The total strain rate is the sum of an elastic and a plastic component,

$$\dot{\epsilon}_{ij} = \dot{\epsilon}_{ij}^e + \dot{\epsilon}_{ij}^p \quad (6.4)$$

The elastic component is related to the stress rate by Hooke's Law,

$$\dot{\epsilon}_{ij}^e = \frac{1}{2G} \dot{\tau}_{ij}' + \frac{1}{3K} \delta_{ij} \dot{\tau} \quad (6.5)$$

The plastic component is given by a strain-rate function,

$$\dot{\epsilon}_{ij}^p = g \frac{\tau_{ij}'}{\bar{\tau}} \quad (6.6)$$

$\bar{\tau}$ is the von Mises effective stress: in terms of the three principal stresses this is given by

$$\bar{\tau} = \sqrt{\frac{1}{2} \left\{ (\tau_1 - \tau_2)^2 + (\tau_2 - \tau_3)^2 + (\tau_3 - \tau_1)^2 \right\}} \quad (6.7)$$

G and K are the elastic shear modulus and bulk modulus, which will be considered functions of the state of the material. The strain-rate function, g, is also a function of the state of the material but primarily dependent on the overstress.

It will be observed that Eq. 6.6 parallels the Reuss equations for quasi-static plastic flow.³⁷ Besides the proportionality of strain rate and deviatoric stress we have tacitly assumed therefore that the principal axes

of strain rate (or of strain increment in time dt) coincide with the principal axes of stress and that the rate of volume change in plastic flow is zero. Furthermore we notice that on squaring both sides of Eq. 6.6 and summing we obtain

$$g = \dot{\bar{\epsilon}}^P \quad (6.8)$$

where $\dot{\bar{\epsilon}}^P$ is formed from $\dot{\epsilon}_{ij}^P$ as $\bar{\tau}$ is formed from τ_{ij} in Eq. 6.7.

We will immediately specialize Eq. 6.6 further by taking g to be a function of the "von Mises effective over-stress," that is to say we write

$$g = g(\bar{\tau}^*) \quad (6.9)$$

where $\bar{\tau}^* = \bar{\tau} - \bar{\sigma}$, and $\underline{\underline{g}}$ satisfies the yield condition. Clearly it will be a convenience in this formulation to use the von Mises yield condition; we will do so, and write the equilibrium yield condition

$$\bar{\sigma} = Y \quad (6.10)$$

The customary geometrical interpretation can be given to these equations. We plot the values of the principal stresses along Cartesian coordinate axes. Equation 6.10, the equilibrium yield condition, is represented by a right circular cylinder of radius $\sqrt{2/3} Y$ whose axis is the line $\sigma_1 = \sigma_2 = \sigma_3$. The stress $\underline{\underline{\tau}}$ is represented by a point on a coaxial cylinder of radius $\sqrt{2/3} \bar{\tau}$. The normal distance from this point on to the equilibrium yield surface is the difference between the radii, $\sqrt{2/3} \bar{\tau}^*$.

An increment of stress $\Delta \underline{\underline{\tau}}$ is represented in this space by a vector which can be resolved into two components parallel to and normal to the cylinder axis. These are the dilatational and deviatoric components of the stress incre-

ment. Equation 6.5 shows that the accompanying elastic strain increment has dilatational and deviatoric components related to the corresponding stress components by the proportionality factors $1/3K$ and $1/2G$, respectively. Thus we may represent the elastic strain increment and its components by the stress increment vector (AC, say) and its components (AB, BC, say), bearing in mind the different proportionality factors for the two components.

The vector AC may also be taken to indicate the change in the plastic strain rate. The plastic strain rate before the stress increment is zero if A lies within the equilibrium yield surface. If A lies outside it, Eqs. 6.6 and 6.9 show that the plastic strain rate is directed along a vector MA, where M is the foot of the normal from A on to the equilibrium yield surface, and its intensity is a function of the length MA. Likewise the plastic strain rate after the stress increment is either zero or is represented by the outward normal NC from the equilibrium yield surface. It may be observed that NC need not contain B since BC may have a component parallel to the yield surface, corresponding to a change in the direction of the plastic strain rate but not of intensity. Its component normal to the yield surface corresponds to a change of intensity of plastic strain rate but not of direction. The vector AB corresponds to a change in hydrostatic pressure and is associated with no change in plastic strain rate.

In this analysis of strain-rate dependent yield behavior we will express the constitutive equation in terms of strain rates and stress rates. We will assume that the values of the elastic moduli G and K , the equilibrium value of Y and the function $g(\bar{\tau}^*)$ can all be described in terms of the state of the material. We postpone for the moment a

discussion of the way in which Y and $g(\bar{\tau}^*)$ vary with the state of the material.

Usually we can presume G , K and g to be functions of the strain and the temperature; \dot{Y} is dependent on the complete history of the material.

Because the conservation equations are written in terms of the total strain, the constitutive equations should also be so written and not left in terms of elastic and plastic strain components. Manipulation of Eqs. 6.4, 6.5, 6.6, and 6.9 gives

$$\dot{\tau}_{ij} = 2G\dot{\epsilon}_{ij} + (K - \frac{2G}{3})\delta_{ij}\dot{\epsilon}_{kk} - 2G g(\bar{\tau}^*) \frac{\tau_{ij}}{\bar{\tau}} \quad (6.11)$$

The equation includes elastic behavior if we write $g(\bar{\tau}^*) = 0$ when $\bar{\tau}^* \leq 0$.

It may be remarked that the neglect in Eq. 6.11 (or in Eqs. 6.4, 6.5, and 6.6) of time derivatives higher than the first implies that we are not concerned with changes in behavior over a time interval as small as that required to accelerate the dislocations, whose movement causes plastic strain, to their terminal velocities. This time is probably of the order of 10^{-11} seconds. In view of the time resolution both of our experimental observations and our computations, the neglect of second and higher order time derivatives is acceptable.³⁸

We will now specialize the description of the strain-rate dependent behavior in three dimensions to the cases of uniaxial stress and uniaxial strain.

For all axially symmetrical situations the principal components of stress and strain may be written (τ_x, τ_t, τ_t) , $(\epsilon_x, \epsilon_t, \epsilon_t)$, respectively, where the subscript x denotes the

axial direction and the subscript t a transverse direction.
It follows that

$$\bar{\tau} = |\tau_x - \tau_t| \quad (6.12)$$

Equation 6.11 becomes in the axial direction,

$$\dot{\tau}_x = (K + \frac{4G}{3}) \dot{\epsilon}_x + (2K - \frac{4G}{3}) \dot{\epsilon}_t - \frac{4G}{3} g(\bar{\tau}^*) \quad (6.13)$$

and in the transverse direction,

$$\dot{\tau}_t = (K - \frac{2G}{3}) \dot{\epsilon}_x + (2K - \frac{2G}{3}) \dot{\epsilon}_t + \frac{2G}{3} g(\bar{\tau}^*) \quad (6.14)$$

To obtain the equation for uniaxial stress, we put $\dot{\tau}_t = 0$
and eliminate $\dot{\epsilon}_t$ between Eqs. 6.13 and 6.14. Recalling that
Young's modulus, $E = 9KG/(3K + G)$, we obtain

$$\dot{\tau}_x = E \dot{\epsilon}_x - \frac{2E}{3} g(\bar{\tau}^*) \quad (6.15)$$

If $Y(\epsilon_x)$ is the yield stress in simple tension for an equivalent work-hardened state,

$$\bar{\tau}^* = \tau_x - Y(\epsilon_x) \quad (6.16)$$

Equations 6.15 and 6.16 describe the strain-rate dependent behavior in uniaxial stress for a given function $g(\bar{\tau}^*)$. It will be observed from Eq. 6.15 that the function g , defined by Eq. 6.6, differs from the function used by Malvern³⁶ by the factor $2E/3$.

The equation in the axial direction for uniaxial strain is obtained immediately from Eq. 6.13 by putting $\dot{\epsilon}_t = 0$.

$$\dot{\tau}_x = (K + \frac{4G}{3}) \dot{\epsilon}_x - \frac{4G}{3} g(\bar{\tau}^*) \quad (6.17)$$

In order to use this equation and retain a one-dimensional description we must express $\bar{\tau}^*$ in terms of τ_x and ϵ_x . This can be done as follows.

In uniaxial strain

$$\dot{\epsilon}_t = \dot{\epsilon}_t^e + \dot{\epsilon}_t^p = 0 \quad (6.18)$$

and since the plastic strain has no spherical component

$$\dot{\epsilon}_x^p + 2\dot{\epsilon}_t^p = 0 \quad (6.19)$$

Therefore

$$\dot{\epsilon}_x = \dot{\epsilon}_x^e + 2\dot{\epsilon}_t^e \quad (6.20)$$

If we now write $\dot{\epsilon}_x^e$ and $\dot{\epsilon}_t^e$ in terms of τ_x and τ_t using Eq. 6.5 we obtain

$$\dot{\epsilon}_x = \frac{1}{3K} (\dot{\tau}_x + 2\dot{\tau}_t) \quad (6.21)$$

This equation may be integrated and rearranged to give

$$\tau_t = \int^t \frac{3K}{2} \dot{\epsilon}_x dt - \frac{1}{2} \tau_x \quad (6.22)$$

Recalling Eq. 6.12 we can now write down the value of $\bar{\tau}^*$, namely

$$\bar{\tau}^* = \frac{3}{2} \left\{ \tau_x - \int^t K \dot{\epsilon}_x dt \right\} - \bar{\sigma} \quad (6.23)$$

and we have achieved our objective.

It may be convenient to express the overstress $\bar{\tau}^*$ in terms of the current state of the material, and this can be done at the expense of introducing an assumption about the nonlinear elastic behavior. We follow Wood³⁹ and Rice et al.¹ and take the value of the integral in Eq. 6.23

at any time to be the value of the pressure $p(\epsilon_x)$ on the hydrodynamic hugoniot curve for the same volumetric strain. The value of $\bar{\tau}^*$ can then be written

$$\bar{\tau}^* = \left| \frac{3}{2} \left\{ \tau_x - p(\epsilon_x) \right\} \right| - \bar{\sigma} \quad (6.24)$$

and if $Y(\epsilon_x)$ is the yield stress in simple tension for an equivalent work-hardened state,

$$\bar{\tau}^* = \left| \frac{3}{2} \left\{ \tau_x - p(\epsilon_x) \right\} \right| - Y(\epsilon_x) \quad (6.25)$$

Equations 6.17 and 6.23 or 6.17 and 6.25 describe the strain-rate dependent behavior in uniaxial stress for a given function $g(\bar{\tau}^*)$.

We will now examine in turn the overstress $\bar{\tau}^*$ and the yield stress $\bar{\sigma}$.

6.2 The Rate-Dependent Overstress

For many engineering purposes the behavior of metals has been described as rate independent, that is to say, $\bar{\tau}^*$ is taken to be zero. This description is also useful in wave propagation analyses since many features of the behavior are not seriously affected if $\bar{\tau}^*$ is neglected. Thus, for example, Bell and his co-workers have been able to show that the von Karman rate-independent theory of wave propagation, when used with the appropriate equilibrium stress-strain relationship, will adequately describe their observations of the symmetrical impact of rods of annealed aluminum⁴⁰ copper⁴¹ and lead.⁴² In uniaxial strain, too, the rate-independent theory has been found to describe many features of the behavior.⁶ However, the rate-independent theory cannot be reconciled with the well-established model for the yield process of dislocation propagation and repositioning

of the atoms in the lattice sites. Plastic flow is not instantaneous. Furthermore, although there is good evidence that a rate effect may be neglected in some situations, there is also good evidence that a rate effect can significantly influence measurable quantities. The evidence for the rate effect in metals has been summarized recently by Johnson, Stein and Davis⁴³ and by Perzyna.⁴⁴

It is therefore desirable to include a rate-dependent overstress in the constitutive relation. However, it is difficult to decide on the functional form of the rate function g (Eq. 6.6). The rate-independent theory may be used to describe most of the wave propagation experiments which have been performed; experimental evidence for the form of g is correspondingly slight. The best that can be done at present is to choose a simple functional form for g which is consistent with current theories of the physical processes taking place and then to use experimental results to put bounds on the values of the coefficients. This has already been attempted. Perzyna^{44,45} uses a formulation of visco-plastic behavior essentially the same as that described above and computes results for a particular experiment of Campbell and Duby using different rate functions. The comparison is inconclusive. At the Los Alamos Scientific Laboratory, rear surface behavior in plate-impact experiments has been calculated, again with a formulation essentially the same as that described above. Only a preliminary account of the work is available at present.⁴⁶ The results support the hypothesis of a rate effect but do not enable a quantitative estimate of the function g to be made with any confidence.

We now develop a simple form for the function g . The discussion closely parallels that of McQueen and Marsh,⁴⁶ but is written in terms of the formulation of mechanical

behavior described above.

Consider an element of material for which the total strain is held constant. Putting $\dot{\epsilon} = 0$ in Eq. 6.11 we obtain for such a condition

$$2G g(\bar{\tau}^*) \frac{\tau'_{ij}}{\bar{\tau}} + \dot{\tau}_{ij} = 0 \quad (6.26)$$

Further, $\dot{\epsilon}_{kk} = 0$ and therefore $\dot{\tau} = 0$ by Eq. 6.5 so that we can write

$$2G g(\bar{\tau}^*) \frac{\tau'_{ij}}{\bar{\tau}} + \dot{\tau}'_{ij} = 0 \quad (6.27)$$

Forming the von Mises effective stress, we get

$$2G g(\bar{\tau}^*) + \frac{\dot{\tau}}{\bar{\tau}} = 0 \quad (6.28)$$

since $\frac{\dot{\tau}}{\bar{\tau}}$ will be negative.

Now we make the assumption that $\frac{\dot{\sigma}}{\sigma} = 0$ so that $\frac{\dot{\tau}}{\bar{\tau}} = \frac{\dot{\tau}^*}{\bar{\tau}^*}$. Certainly if the rate effect is significant the strain hardening will be small compared with the stress relaxation for a given amount of plastic flow. Equation 6.28 can then be rewritten as

$$-\frac{\bar{\tau}^*}{\bar{\tau}} \frac{dt}{d\bar{\tau}^*} = \frac{\bar{\tau}^*}{2G g(\bar{\tau}^*)} \quad (6.29)$$

which gives the time required to relax the overstress at the current rate. This "relaxation time" we will write as $h(\bar{\tau}^*)$.

To the extent that plastic flow is due to the propagation of dislocations, it is a thermally activated process and follows Boltzmann's Law. With this tenuous justification we assume that h may be expressed

$$h(\bar{\tau}^*) = H \exp \left\{ \frac{U - c\bar{\tau}^*}{kT} \right\} \quad (6.30)$$

where H is a constant with dimensions of time, U is an activation energy, c is constant with dimensions of volume, k is Boltzmann's constant and T is the absolute temperature.

We then have

$$g(\bar{\tau}^*) = a \bar{\tau}^* e^{b \bar{\tau}^*} \quad (6.31)$$

where $a = 1/(2GHe^{U/kT})$ with dimensions $(\text{stress} \cdot \text{time})^{-1}$ and $b = c/kT$ with dimensions $(\text{stress})^{-1}$. This is the form for g used in Sect. 6.4. The constants a and b cannot be determined from basic physical constants; their values must be found empirically. The formula may be expected to fit experimental data only when the conditions are such that the plastic flow is predominantly thermally activated.

It may be inferred from the work of Dorn and his collaborators^{38,47} that at low temperatures (less than about 200 °K for aluminum) or at high strain rates (above 10^5 sec^{-1} for aluminum) g will assume some other form, more nearly linearly related to stress and temperature. This hypothesis is supported by the observation that plastic deformation by twinning is observed both at low temperatures and at high strain rates, which shows that the flow mechanisms are indeed different from those operative under less extreme conditions.

It is interesting to calculate approximately the effect of temperature changes on the function g in the form given by Eq. 6.31. In an ordinary tensile test at room temperature when plastic flow is easily observable, the value of $\dot{\epsilon}^P$ is of the order of 10^{-3} sec^{-1} and the value of $(U - c\bar{\tau}^*)$ must be about 1 eV. The strain rates we are interested in

are more like 10^4 sec^{-1} . For this change of rate at the same temperature, $(U - c\bar{\tau}^*)$ must be reduced to about 0.6 eV. Let us say then that the energy to be supplied by thermal fluctuations is 0.6 eV. If this is the case, the strain rate doubles for a 9°C rise in temperature from 293°K to 302°K , the overstress being held constant. Alternatively the strain rate can be maintained while the temperature increases 9°C if the value of $c\bar{\tau}^*$ is reduced from about 0.40 to about 0.38. The results of Hauser, Simmons and Dorn⁴⁷ indicate that for aluminum, c has a value of the order of 1 eV/kb. The stress change required to maintain the strain rate is therefore of the order of 0.02 kb.

We cannot carry this analysis of the temperature dependence of g very far, since the functional form of g is only a conjecture. However, it serves to show how strongly g is dependent upon temperature when the process is thermally activated. It suggests, too, that the strain-rate dependent overstress may be negligible in uniaxial-strain wave propagation when the plastic flow is thermally activated, because of the temperature rise through the loading shock wave. For aluminum, a modest pressure rise of, say, 100 kb is associated with a temperature rise of about 74°C .¹

On the other hand, stress activated processes, which are not strongly temperature dependent, are important if, as seems likely, they control the behavior at the higher strain rates.

6.3 The Yield Condition

We now examine the problem of determining the yield stress in terms of known measures of the state of the material.

The yield condition can be defined in different ways. The static yield condition gives the combination of stresses

at which plastic flow is first observable if the material is slowly loaded from an elastic condition. The dynamic yield condition for a material which does not show a strain-rate effect gives the combination of stresses required for plastic flow to take place in plastic wave propagation. These two yield conditions may differ. For annealed copper for example, Bell and Werner have shown the static and dynamic stress-strain curves in uniaxial stress to be significantly different.⁴¹ Here the dynamic curve is the relationship between stress and strain which when used with the von Karman theory of plastic wave propagation gives results which agree with experiment.

One would expect the static and dynamic yield conditions to be the same. Both can be defined in terms of Eq. 6.2 as the stress $\underline{\sigma}$ associated with an infinitesimal plastic strain rate. That they are not coincident, as determined from static and dynamic experiments, might well be due to time-dependent processes taking place in the material at yield stress levels. In the time taken to conduct a static tensile test there may be, for example, ageing and self-annealing mechanisms active which in the time of the dynamic experiment are effectively inoperative. Therefore, if the yield condition is required for use in wave propagation analysis, it should be determined from wave-propagation experiments. It is then the yield condition appropriate for the mechanisms which are operative in the case being analyzed.

Such a dynamic determination of the yield condition is not easily made. It may be inferred from the previous discussion that the Hugoniot yield point is not determinable from the records of measurable quantities unless the rate function g is known. The determination of the yield condition for a particular material calls, therefore, for a carefully designed program of experimentation and analysis.

Very few experiments of this kind have been done. Probably the only series of experiments from which a yield condition can be deduced with confidence is the work of Bell^{40,41} which is confined, however, to uniaxial stress conditions and to pure, dead-annealed material. The work of Dorn and his collaborators³⁸ is also thoroughly done and covers hardened materials but is once again confined to supposedly uniaxial stress conditions. Dorn's results are deduced from Hopkinson bar measurements of the behavior of the ends of the specimens; the analysis assumes that the stress is in fact uniaxial and that the material behavior can be inferred from average values of the variables over short time intervals. Until a proper axially symmetric two-dimensional analysis of inelastic wave propagation in rods is available it will not be possible to decide to what extent the assumptions made in the analysis invalidate the results. Other work, similar to that of Dorn et al., has been done by Ripperger and others at the University of Texas.⁴⁸ They have attempted to account for both the strain-rate effect and the radial stresses in their analysis but the results are not conclusive.

Although experiments on wires or rods are in many ways attractive, they suffer from the serious disadvantage that the stress is not in fact uniaxial. Without careful analysis of the complete two-dimensional wave system, which the experiment was originally designed to avoid, it is not possible to isolate the effects of various secondary features of the behavior, such as strain-rate effects, nonplanar wave fronts and radial motion. This criticism does not apply to plate experiments where the wave propagation is in uniaxial strain. However, they are not easily carried out and very little use has been made of plate impact or other longitudinal plane wave experiments to investigate yield behavior. Both Fowles,⁴⁹ who sacrificed the simplicity of a purely uniaxial geometry in

order to observe the behavior on a wedge surface, and Lundergan⁵⁰ have determined the Hugoniot elastic limit for a particular aluminum alloy at room temperature. No other conclusive determination of the dynamic yield condition for a metal in uniaxial strain has been published.

Information on the yield condition from the theoretical point of view is also sparse. There is no lack of theories to describe details of the behavior but it is not possible at present to put together a quantitative description of the yield condition. If the yield condition is to be in the form of a "mechanical equation of state," $\underline{\sigma} = f(\underline{\epsilon}, T)$, where T is the temperature, then it must follow that the structure of the material is preserved in the plastic flow. This is known not to be so. The structure of the material is changed by deformation and the functional dependence of the yield stress on strain and temperature changes with it. If the structure of the material could be described by a number of parameters $S_1, S_2 \dots S_n$ then it should be possible in theory to write $\underline{\sigma} = f(S_1, S_2 \dots S_n, T)$ as the yield condition. Two parameters of this kind which have been used are the total strain $\underline{\epsilon}$, and the total plastic work, but the reason for their choice appears to be mathematical convenience rather than physical realism. It can be shown experimentally that the yield condition depends upon at least two independent structural parameters.⁵¹ In fact one might expect more, for there can be several mechanisms contributing to the hardening, and each behaves in its own way with changing temperature and changing plastic strain, depending on the annealing, diffusion and other processes taking place.

The yield condition, and its dependence on the state of the material therefore is not known and it is not possible to formulate a hypothetical yield condition in such a way that

the complex dependence on all of the previously-mentioned factors is included in a physically realistic way.

The problem remains of what value to give to g in a wave propagation analysis. It is evidently necessary to make some arbitrary assumptions, and the most convenient assumption is that, for a von Mises yield condition, $\bar{\sigma}$ is a material constant. Although the functional dependence of $\bar{\sigma}$ on the state variables is not known, this elastic-plastic assumption can be improved. For example, the strain hardening may be included by taking the value of $\bar{\sigma}$ from a tensile test at a strain which represents an equivalent amount of total plastic work.⁴⁹ Again, it is an improvement over ignoring the temperature effect to assume that the value of $\bar{\sigma}$ decreases to zero at the melting point.

Whether or not such arbitrary phenomenological descriptions of the behavior are adequate for engineering purposes depends not so much on their physical exactness as on the effect of their inexactness on the quantities of interest in a particular configuration. If modifications to the yield behavior have only a very small effect on quantities of interest for a certain class of engineering problems, then for the analysis of this class of problems the yield behavior may be prescribed only approximately. In the next section we investigate the effect of some simple descriptions of yield behavior for a particular example of wave propagation in uniaxial strain.

6.4 Calculations of Wave Propagation in Uniaxial Strain

The discussions of the two preceding sections can be put into perspective by considering some special cases, and calculating the effect on observable quantities of changes in the yield behavior. We have chosen to calculate the be-

havior of an impacted aluminum plate at two different impact velocities, for various modifications of the simple elastic-plastic behavior.

There are of course a great many configurations and different materials for which the observable behavior might usefully be calculated. A complete appreciation of the relative importance of the modifications to the yield behavior will not be obtained from calculations of only one configuration. The work of this section should therefore be regarded as only a beginning to a proper investigation.

Calculations are made for the configuration shown diagrammatically in figure 6.1 for two different velocities of the projectile, called Cases A (velocity v) and B (velocity $3v$). The calculations have been made using Wave II, a one-dimensional code for wave propagation in a solid with an arbitrary constitutive equation, which is described in a report by Herrmann and Mack.⁵² Its accuracy, especially for the strain-rate case, is demonstrated in the report. Four different types of yield behavior are considered. Elastic-plastic with a constant yield point, that is to say perfectly plastic; (Cases A1, B1) elastic-plastic with a pressure-dependent yield point (Cases A2, B2); elastic-plastic with strain hardening (Cases A3, B3); elastic-plastic with a strain-rate effect (Cases A4.1, A4.2, B4.1, B4.2).

The constitutive equations chosen for calculation are nominally for aluminum. The basic elastic-plastic behavior is taken to follow the Wood theory with the Murnaghan expression for the hydrodynamic Hugoniot, the constants being taken from static data, which have been shown to give good results for initial loading in the case of 6061-T6 alloy.⁶ The modifications to this elastic-plastic behavior are arbitrary, however, and are not intended to represent the behavior of any

particular real material. The actual constitutive relations chosen for the different cases are now described.

Case 1. The elastic-plastic behavior is intended as an approximation to the behavior of 6061-T6 aluminum, and is taken as follows:

$$\text{Elastic} \quad d\tau/dv = -1067.4 \text{ kb}$$

$$\text{Plastic loading} \quad \tau = p + 1.84 \text{ kb}$$

$$\text{Plastic unloading} \quad \tau = p - 1.84 \text{ kb}$$

where $p = 169.38 (v^{-4.34} - 1) \text{ kb}$

τ is the axial stress

v is the ratio ρ_0/ρ

ρ is the density and ρ_0 its initial value

p is the hydrodynamic hugoniot pressure for the density ρ .

Case 2. Under ordinary engineering conditions it is well established that the effect of hydrostatic pressure on the yield behavior is very small indeed. It is usually presumed that the same is true for dynamic behavior in uniaxial strain. This may be so, but it is nevertheless of interest to see what would be the effect on observable quantities of a pressure-dependence of the yield point. Calculations of Case 2 are intended to show this. The pressure dependence assumed is that the value of Y , the uniaxial yield stress, (which is 2.76 kb in the initial state) is increased by 0.03 kb for every 1 kb rise in dilatational stress, or mean pressure. The only change, then, from the equations of Case 1 is as follows:

$$\text{Plastic loading} \quad \tau = 1.02p + 1.84 \text{ kb}$$

$$\text{Plastic unloading} \quad \tau = 0.98p - 1.84 \text{ kb}$$

Case 3. It is argued in Sect. 6.3 that it is not possible to describe the strain hardening of a metal with a single parameter. However, the effects of strain hardening may be judged (since they prove to be small) by taking the hardening, or increase in the yield stress, to be a function of the total plastic work. We have in this case arbitrarily assumed the increase in the value of Y to be proportional to the total plastic work per unit volume W_p .

The formulation of the constitutive relations follows the theory set out by Fowles.⁴⁹ We write

$$dW_p = \frac{2}{3} Y(d\epsilon_x - \frac{1}{2G} dY)$$

and

$$dY = \eta dW_p$$

where ϵ_x is the axial strain and G is the shear modulus. The following constitutive equations result.

$$\text{Plastic loading} \quad \tau = p + 2/3Y$$

$$\text{Plastic unloading} \quad \tau = p - 2/3Y$$

where in the finite difference formulation Y is calculated thus:

If a mesh point is elastic at time $n\Delta t$, $Y^{n+1} = Y^n$

If a mesh point is plastic and loading at time $n\Delta t$,

$$Y^{n+1} = Y^n + \frac{2Y^n(\epsilon^{n+1} - \epsilon^n)}{\frac{3}{\eta} - (\epsilon^{n+1} - \epsilon^n) + \frac{1}{G} Y^n}$$

If a mesh point is plastic and unloading at time $n\Delta t$,

$$Y^{n+1} = Y^n + \frac{2Y^n(\epsilon^n - \epsilon^{n+1})}{\frac{3}{\eta} - (\epsilon^n - \epsilon^{n+1}) + \frac{1}{G} Y^n}$$

We took

$$Y^0 = 2.86 \text{ kb}$$

$$G = 249.2 \text{ kb}$$

$$\eta = 10.0$$

The stress-strain curve in simple tension for this hardening law is shown in figure 6.2.

Cases 4.1, 4.2. The way in which a strain-rate effect can be included in the constitutive equations is described in Sect. 6.1. For reasons set out in Sect. 6.2, we chose a strain-rate function as follows:

$$g = a \bar{\tau}^* \exp (b \bar{\tau}^*)$$

where $\bar{\tau}^*$ is the von Mises effective overstress. It is not possible from data presently available to put values on the constants a and b for aluminum. We expected, however, that a value for g of the right order of magnitude would be obtained by taking $a = 10^3 \text{ kb}^{-1} \text{ sec}^{-1}$ and $b = 5.0 \text{ kb}^{-1}$.

Unfortunately we have not yet been able to run a calculation satisfactorily with these values for a and b , as they lead to stress relaxation rates which cannot be assumed constant during the smallest time increment for the calculation which can reasonably be used. Until the machine program has been modified to take care of this, only values for g that give relaxation rates which are reasonably constant in the time increment used for calculation can be accommodated. We have run calculations for two different strain-rate func-

tions, which satisfy this requirement.

In Case 4.1 we took $a = 100 \text{ kb}^{-1} \text{ sec}^{-1}$
 $b = 0.5 \text{ kb}^{-1}$

In Case 4.2 we took $a = 300 \text{ kb}^{-1} \text{ sec}^{-1}$
 $b = 1.5 \text{ kb}^{-1}$

The overstress $\bar{\tau}^*$ is calculated thus:

If $Y < \frac{3}{2} (\tau - p)$, $\bar{\tau}^* = \frac{3}{2} (\tau - p) - Y$

If $-Y \leq \frac{3}{2} (\tau - p) \leq Y$, $\bar{\tau}^* = 0$

If $\frac{3}{2} (\tau - p) < -Y$, $\bar{\tau}^* = \frac{3}{2} (\tau - p) + Y$

It should be emphasized that these cases are not intended to represent the behavior of a real material. We can, however, expect the behavior of aluminum to lie between Case 4.2 and the strain-rate independent Case 1.

For all cases, the fracture stress was set at -10.0 kb . As will be seen from the results, Case B fractured and Case A did not, except for Case A4.1. Also, in all cases an artificial viscosity was introduced to ensure stability and smoothness for the finite-difference machine calculation described in the Wave II report.⁵² The values used for the coefficients described therein were $A_1 = 2.0$, $B_1 = 0.125$, $A_2 = 0$, $B_2 = 0$.

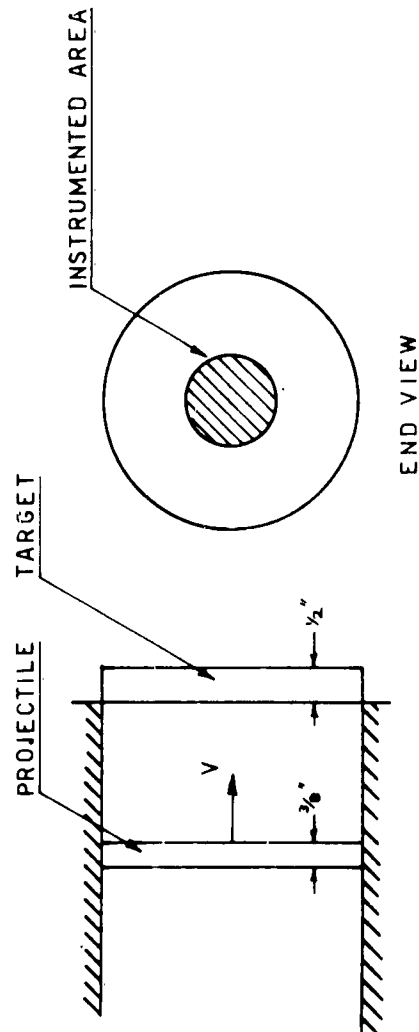
From the machine output, three quantities have been plotted against time; the interface stress, the rear surface position, and the rear surface velocity. Representative re-

sults are displayed in figures 6.3 through 6.9. It may be of help in the interpretation of these figures to refer to the characteristic diagram figure 6.10. This Lagrangian characteristic mesh in the $x - t$ plane shows the wave fronts for Case A and a linearized elastic-plastic behavior.⁶

The conclusion to be drawn from a study of the results of the calculations is that none of the measurable quantities, impact face stress or rear surface velocity or position, differ sufficiently in the different cases to provide a measure of the effects on yielding which we are considering.

A preliminary comparison is possible with data provided by C. D. Lundergan⁵³ of Sandia Corporation. The measured rear-surface displacement vs. time is plotted in figure 6.11 where it is compared with the calculation for Case A1. It may also be compared with figure 6.3 and 6.4. It is clear that an experiment of this type is not suitable to resolve between the behavior assumed for Cases A1, A2, A3, A4.1 or A4.2.

From the information in this section, it may be concluded that the values of yield stress and rate-dependent over-stress may be determined only very approximately from uniaxial strain experiments, which provide sufficiently simple boundary conditions to permit solutions to be obtained. In one respect this is encouraging, for it means that the behavior can be adequately described by a simple rate independent elastic-plastic constitutive relation for cases of uniaxial strain at low stresses. On the other hand, the rate independent description is expected to be inadequate on physical grounds. For other configurations, the simple rate-independent theory may be inadequate, but solution methods must be developed for other configurations before they can be investigated.



CASE A $V = 4.961 \times 10^3$ in/sec
CASE B $V = 14.883 \times 10^3$ in/sec

Figure 6.1. Diagrammatic representation of the plate impact experiment which was analyzed.

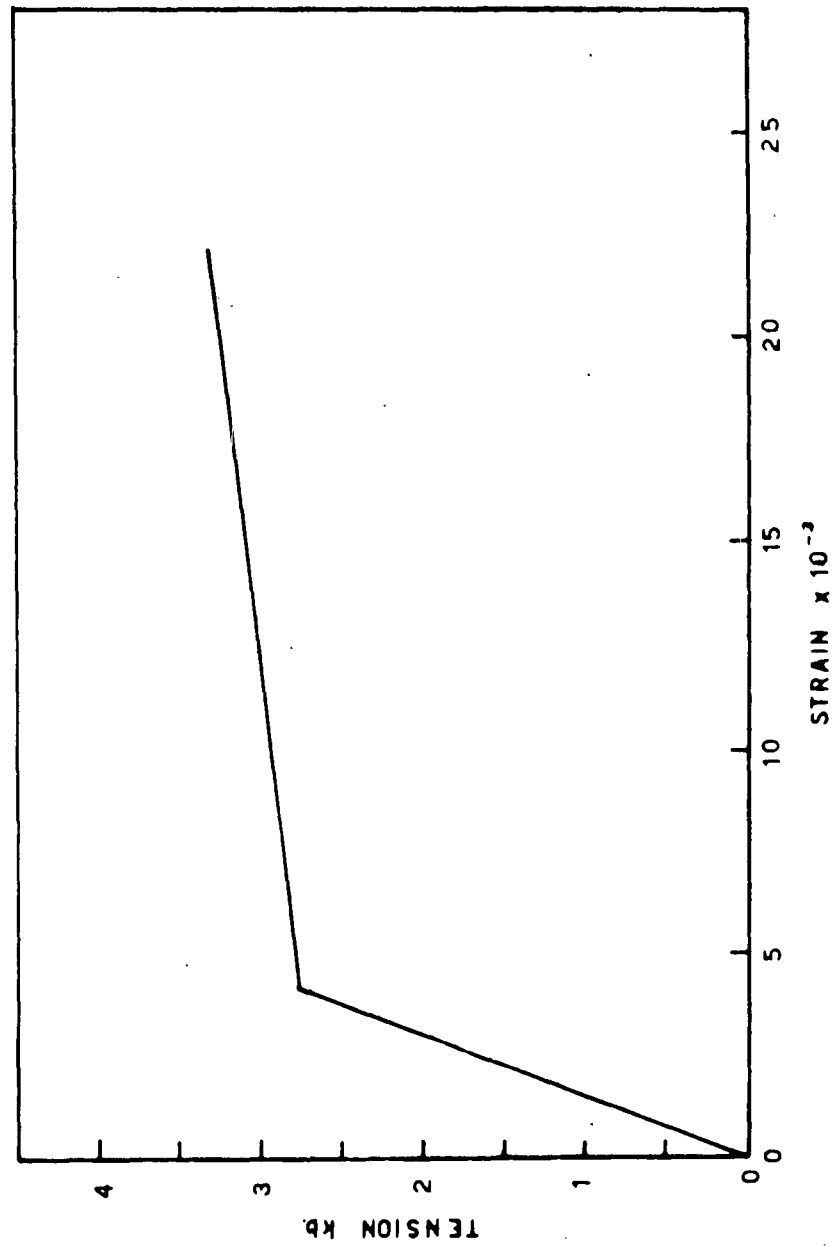


Figure 6.2. The stress-strain relation in simple tension for the strain-hardening behavior of Case 3.

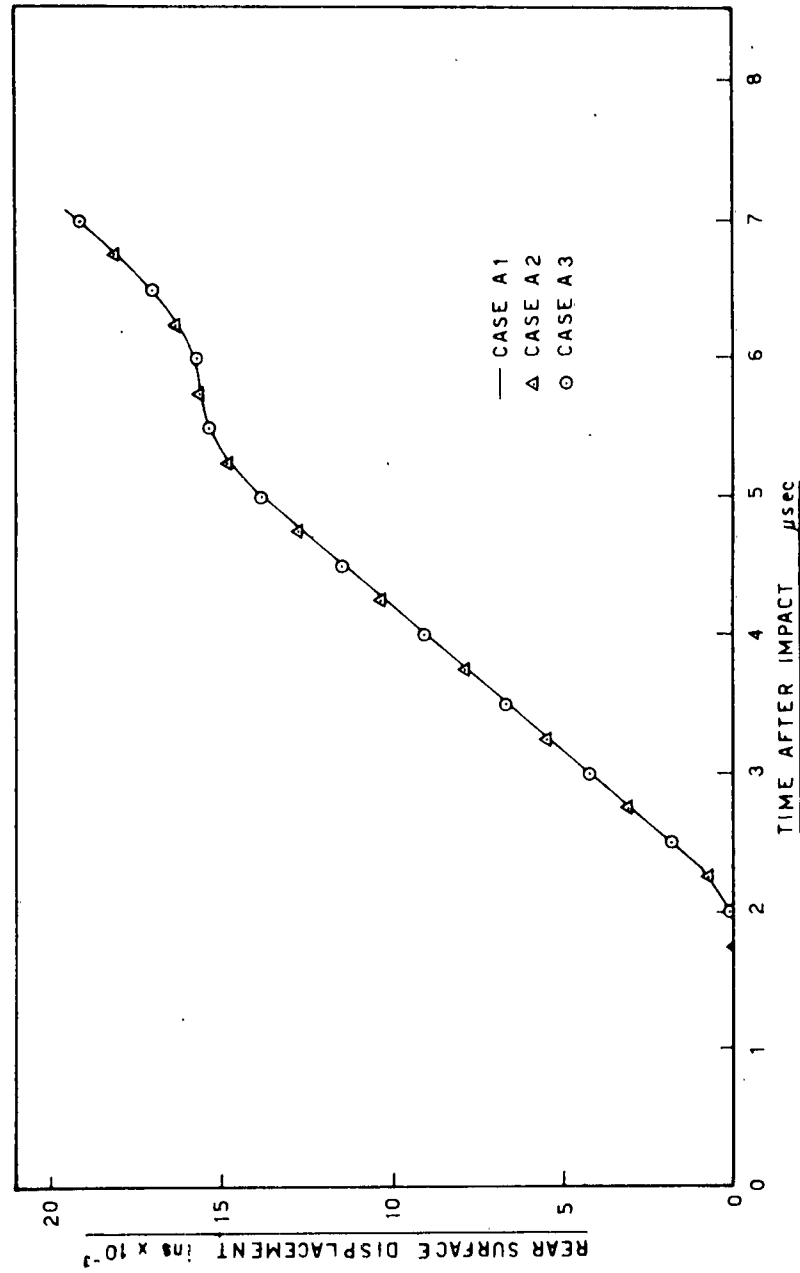


Figure 6.3. Rear surface displacement for Cases A1, A2 and A3.

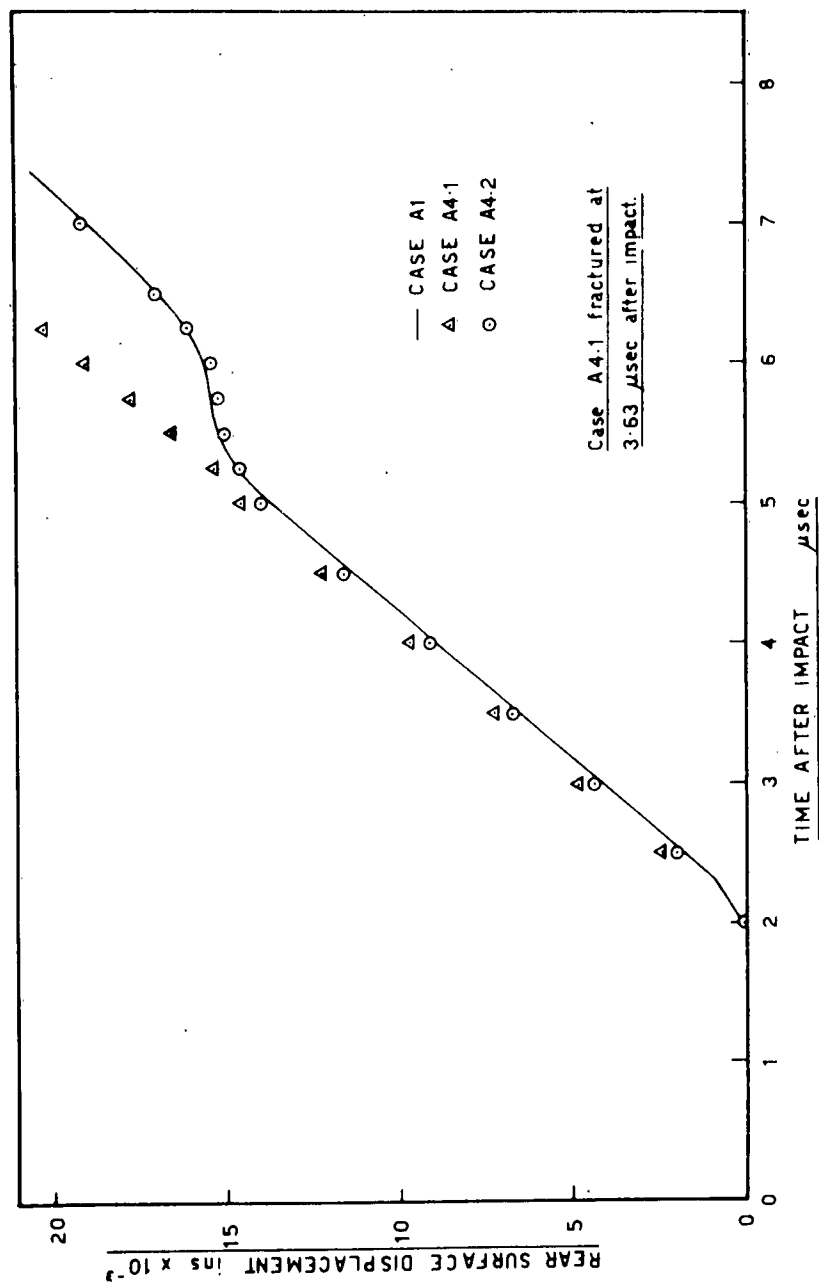


Figure 6.4. Rear surface displacement for Cases A1, A4.1 and A4.2.

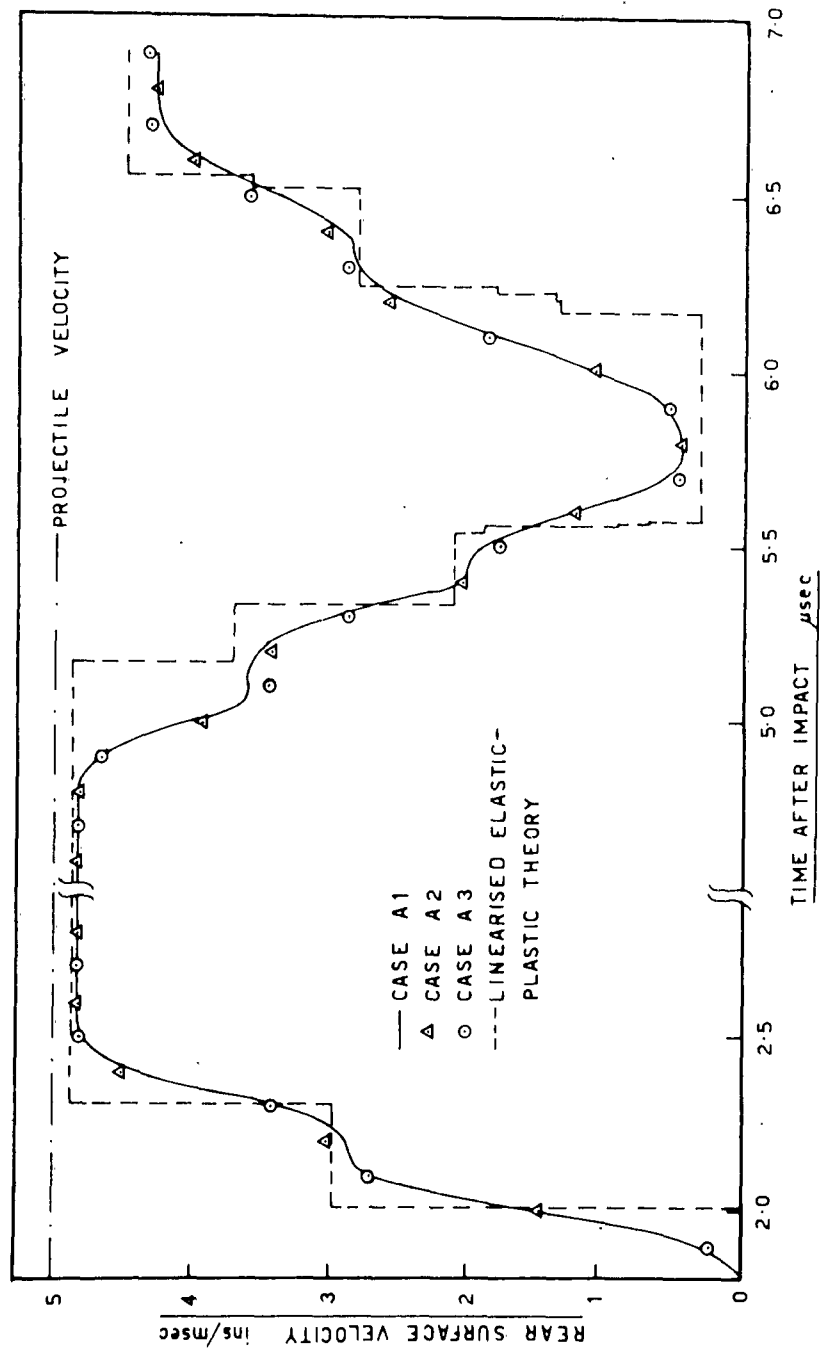


Figure 6.5. Rear surface velocity for Cases A1, A2 and A3.

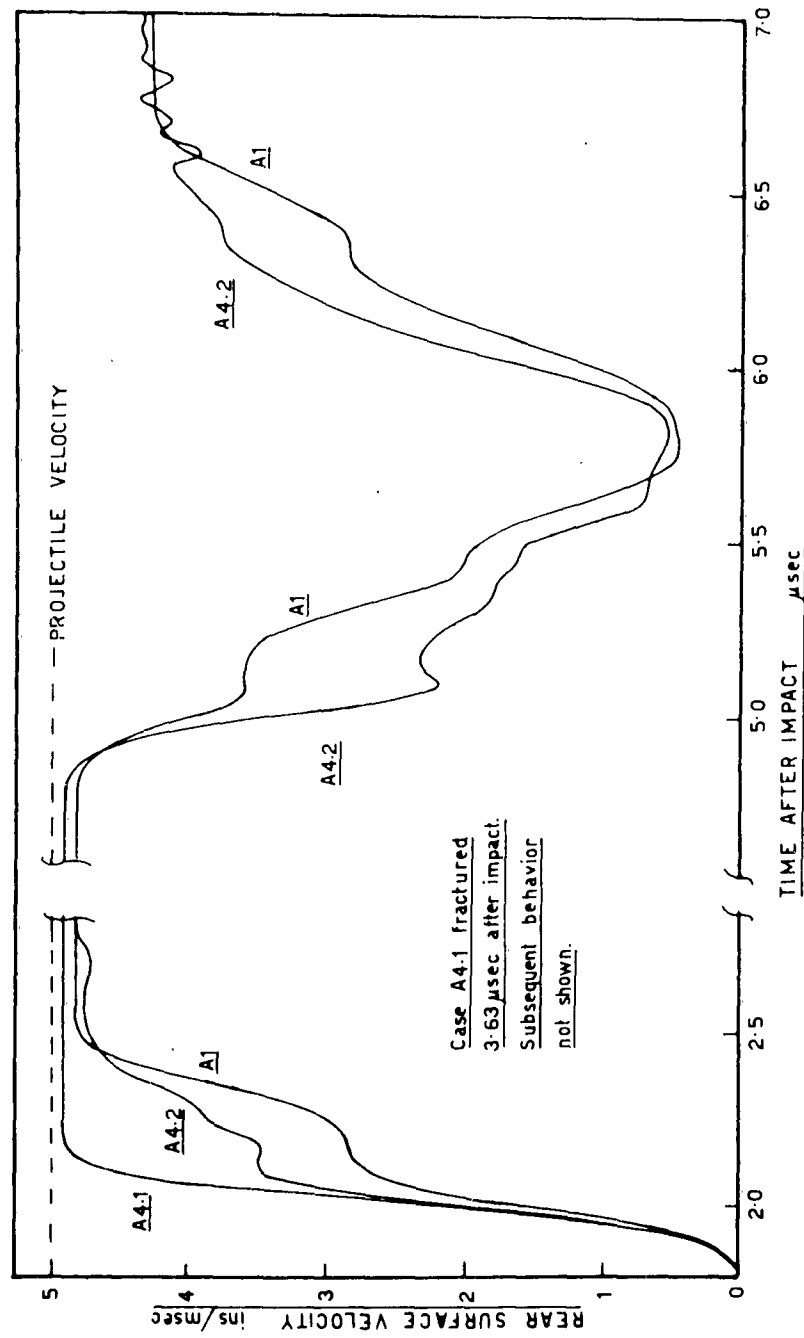


Figure 6.6. Rear surface velocity for Cases A1, A4.1 and A4.2.

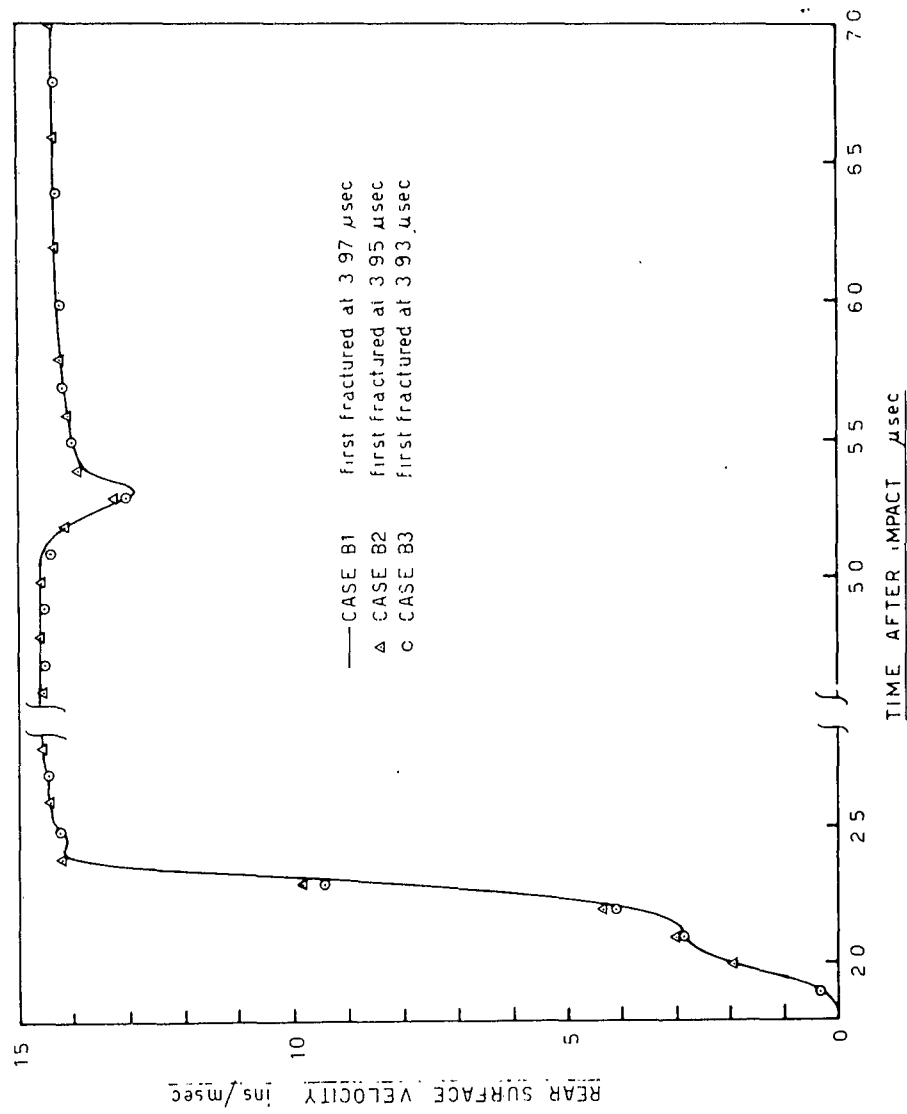


Figure 6.7. Rear surface velocity for Cases B1, B2 and B3.

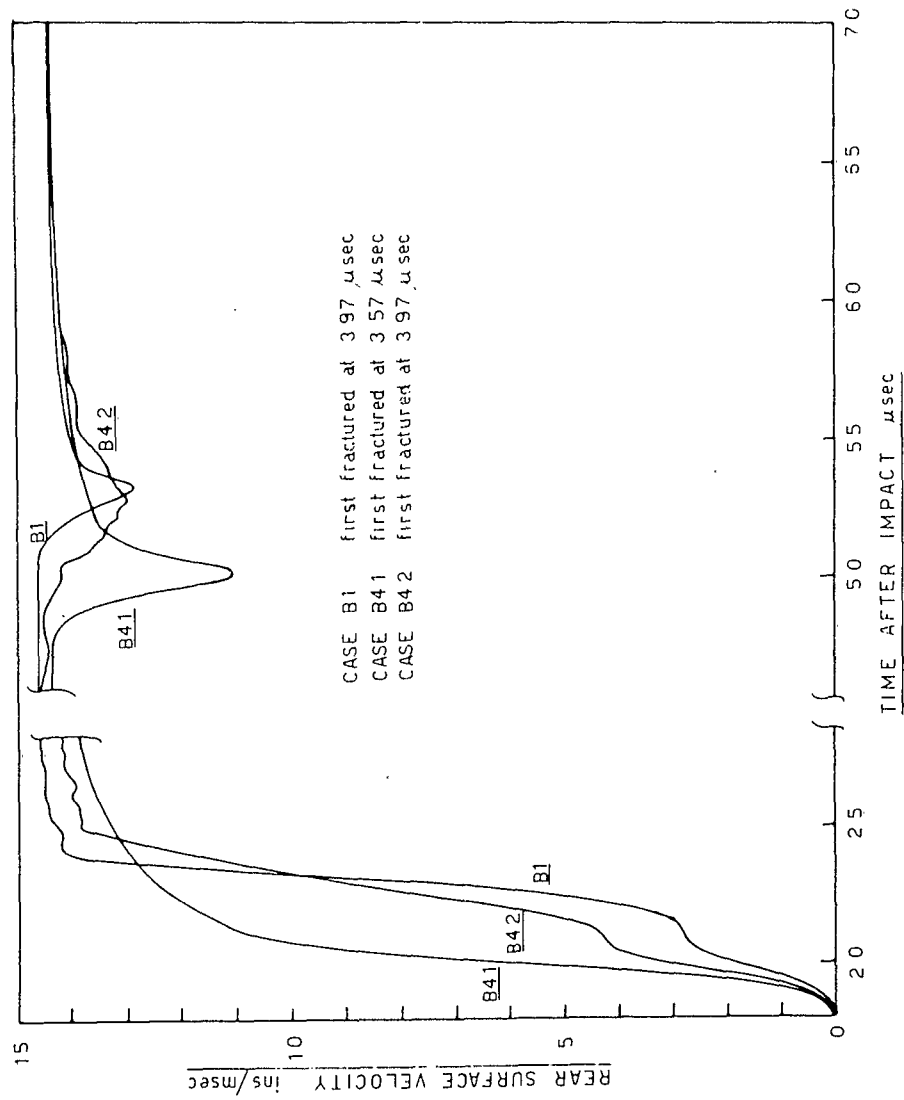


Figure 6.8. Rear surface velocity for Cases B1, B4.1 and B4.2.

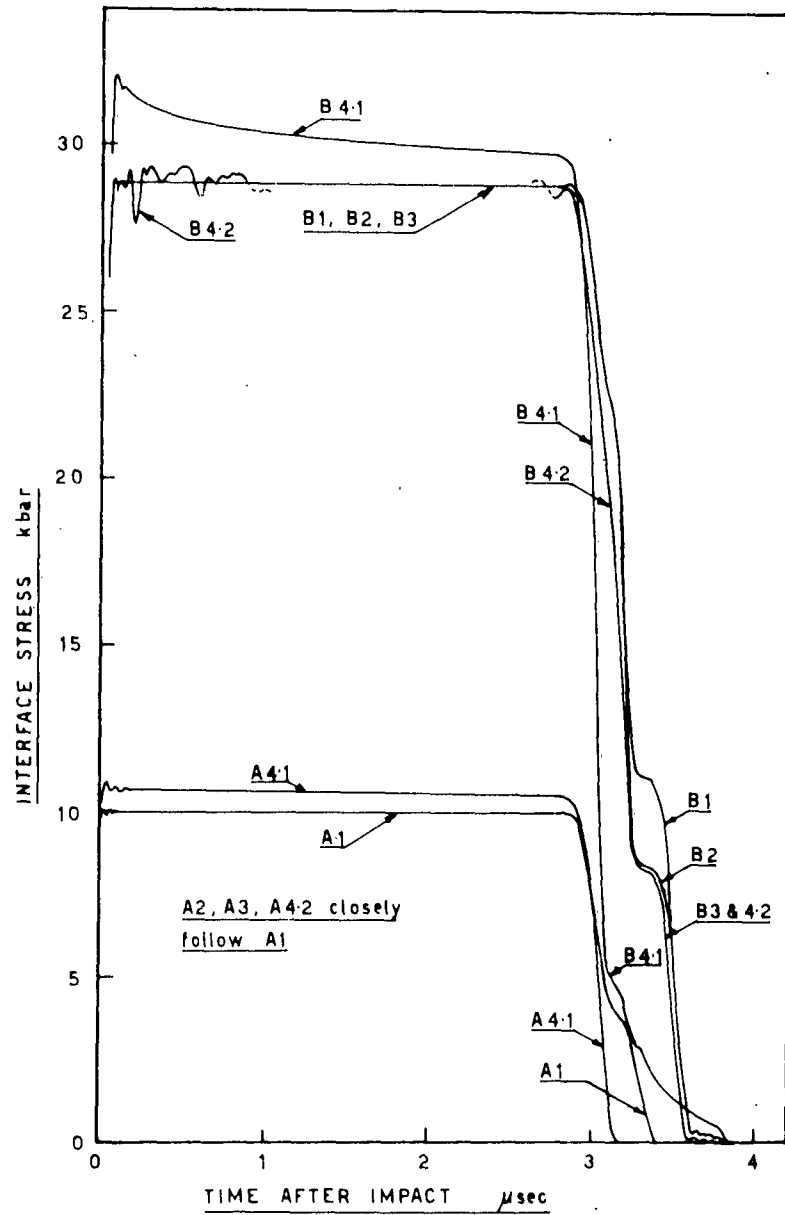


Figure 6.9. Impact face stress for all Cases A and B.

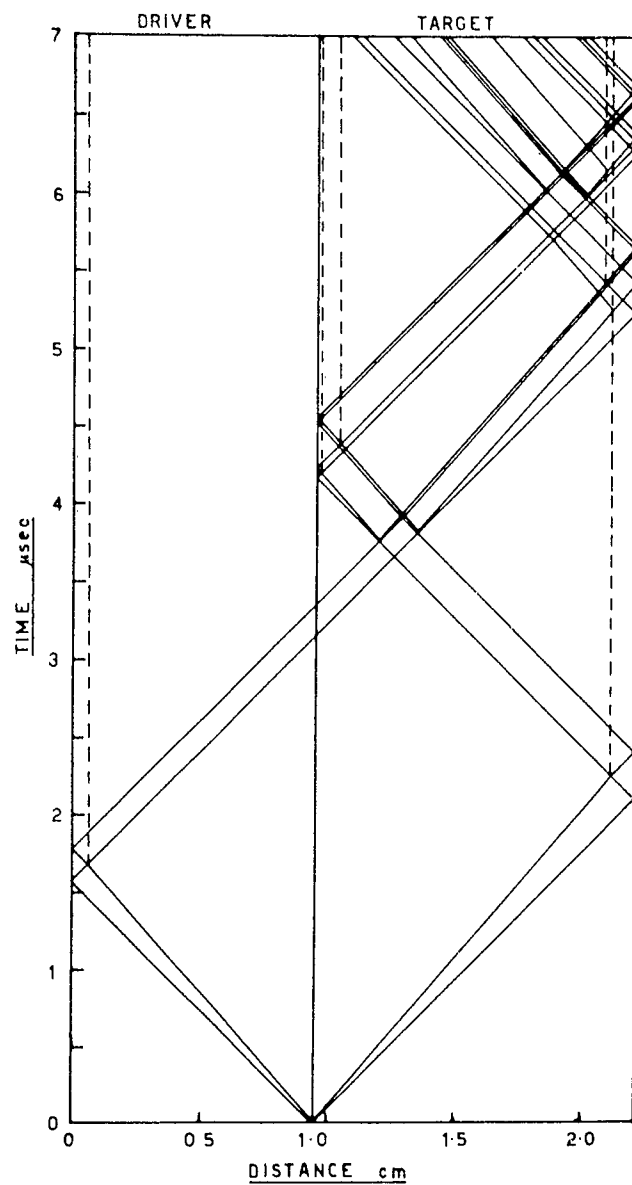


Figure 6.10. Lagrangian characteristic diagram showing the elastic and plastic wave fronts for Case A and linearized elastic-plastic behavior.

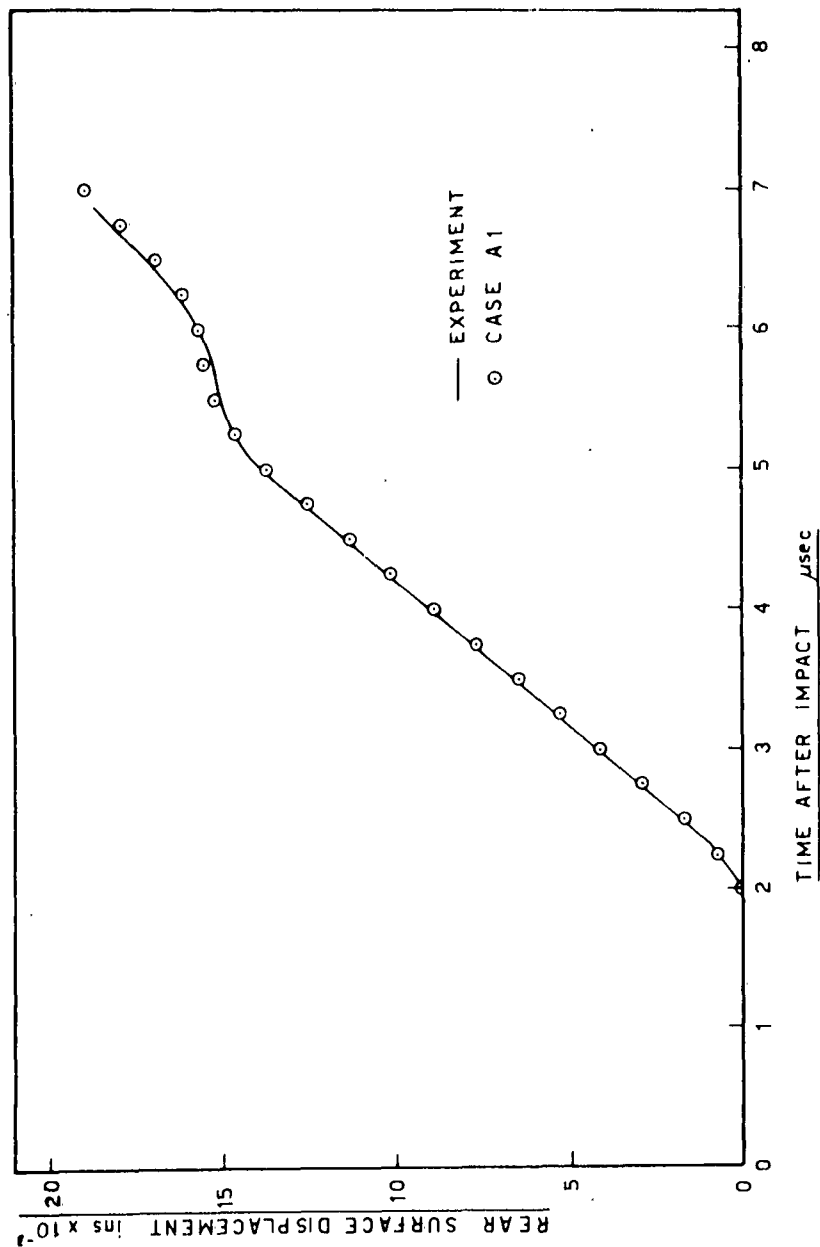


Figure 6.11. Rear surface displacement of Case A observed by C. D. Lundergan⁵³ compared with calculated Case A1.

REFERENCES

1. Rice, M. H., McQueen, R. G., and Walsh, J.M. "Compression of Solids by Strong Shock Waves." Solid State Physics, Editors: Seitz and Turnbull, Vol. 6, pp 1-63, 1958.
2. Brode, H. Dynamic Properties of Matter under High Stress. Thermodynamic Description. Symposium on Structural Dynamics under High Impulsive Loading, Dayton, Ohio. September 1962 (to be published).
3. Prager, W. Introduction to Mechanics of Continua. Ginn and Company, Boston, 1961.
4. Courant, R. and Friedrichs, K. O. Supersonic Flow and Shock Waves. Interscience Publishers, Inc., New York, 1948.
5. von Karman, T. On the Propagation of Plastic Deformation in Solids. Office of Scientific Research and Development, OSRD 365, 1942.
6. Herrmann, W., Witmer, E. A., Percy, J. H., and Jones, A. H. Stress Wave Propagation and Spallation in Uniaxial Strain. ASD Report TDR 62-399(1962). Contract No. AF 33(616)-6373.
7. Rich, J. C. ed. AFSWC Second Hydrodynamic Conference. Numerical Methods of Fluid Flow Problems. AFSWC-TN-61-29, Part 1, May 1961.
8. Richtmyer, R. D. Difference Methods for Initial-Value Problems. Interscience Publishers, Inc., New York 1957.

REFERENCES Continued

9. Harlow, F. H. and Meixner, B. D. The Particle and Force Computing Method for Fluid Dynamics. Los Alamos Scientific Laboratory, LAMS-2567, October 1961.
10. Kolsky, H. G. The Nearest Neighbor Hydrodynamic Calculation. Los Alamos Scientific Laboratory, Report LASL T-S, July 1, 1961.
11. Amurud, L. and Orr, S. R. A Note on Inverted Centers of Pressure and Crossed Mass Points in a Two-Dimensional Hydrodynamic Calculation. Los Alamos Scientific Laboratory, May 26, 1958.
12. Truesdell, C. and Toupin, R. A. "The Classical Field Theories." Encyclopedia of Physics, Edited by S. Flügge, Vol. 3/1, p. 226-793, 1960.
13. Freudenthal, A. M. and Geringer, H. "The Mathematical Theories of the Inelastic Continuum." Encyclopedia of Physics, Edited by S. Flügge, Vol. 6, p 229-433, 1958.
14. DeGroot, S. R. and Mazur, P. Non-Equilibrium Thermodynamics. North Holland Publishing Co. Amsterdam, 1962.
15. Ziegler, H. "An Attempt to Generalize Onsager's Principle and its Significance to Rheological Problems." Z.A.M.P. Vol. 9b, 1958, p 748-763.
16. Eckart, Carl. The Thermodynamics of Irreversible Processes IV The Theory of Elasticity and Anelasticity. Phys. Rev. 73 373 (1948).
17. Vakulenko, A. A. "The Connection between Stresses and Strains in Inelastic Media." Soviet Physics Doklady, Vol. 3, 1958, pp. 193-196.

REFERENCES Continued

18. Vakulenko, A. A. "A Thermodynamic Investigation into the Relation between Stress and Deformation in Isotropic Elastoplastic Media." Soviet Physics Doklady, Vol. 4, 1959, pp. 697-700.
19. Backman, M. E. Plasticity and Non-linear Elastic Strains. NAVWEPS Report 7648, NOTS TP 2665, 27 April 1961.
20. Murnaghan, F. D. Finite Deformation of an Elastic Solid. J. Wiley and Sons, Inc., New York, 1951.
21. Birch, F. "The Effect of Pressure upon the Elastic Parameters of Isotropic Solids, According to Murnaghan's Theory of Finite Strain." J. of Applied Physics, Vol. 9, 1938, p. 279-288.
22. Toupin, R. A. and Bernstein, B. "Sound Waves in Deformed Perfectly Elastic Materials; the Acoustoelastic Effect." J. Acoust. Soc. Amer., Vol. 33, 1961, pp. 216-225.
23. Hayes, M. and Rivlin, R. S. "Propagation of a Plane Wave in an Isotropic Elastic Material Subject to Pure Homogeneous Deformation." Arch Rational Mech. Anal., Vol. 8, 1961, pp. 15-22.
24. Huntington, H. B. "The Elastic Constants of Crystals." Solid State Physics; Editors Seitz and Turnbull; Vol. 7, pp. 213-351, 1958.
25. Hearmon, R. F. S. "The Elastic Constants of Anisotropic Materials II." Adv. in Physics, Vol. 5, 1956, pp. 323-382.
26. Lazarus, D. "The Variation of the Adiabatic Elastic Constants of KCl, NaCl, CuZn, Cu, and Al with Pressure to 10,000 Bars." Phys. Rev., Vol. 76, 1949, pp. 545-553.

REFERENCES Continued

27. Daniels, W. B. and Smith, C. S. "Pressure Derivatives of the Elastic Constants of Copper, Silver and Gold to 10,000 Bars." Physical Review, Vol. 111, 1958, pp. 713-721.
28. Hughes, D. S. and Maurette, C. "Dynamic Elastic Moduli of Iron, Aluminum, and Fused Quartz." J. of Applied Physics, Vol. 27, 1956, pp. 1184-1186.
29. Corll, J. A. "Pressure Derivatives of the Elastic Constants of Cadmium." Case Institute of Technology, ONR Technical Report No. 6, June 1962.
30. Hughes, D. S. and Kelly, J. L. "Second-Order Elastic Deformation of Solids." Physical Review, Vol. 92, 1953, pp. 1145-1149.
31. McQueen, R. G. and Marsh, S. P. One-Dimensional Compression Waves in Isotropic Solids. Los Alamos Scientific Laboratory. Unpublished report, 1961.
32. Altshuler, L. V., Kormer, S. B., Bakanova, A. A., and Trunin, R. F. "Equation of State of Aluminum, Copper and Lead in the High Pressure Region." Soviet Physics. JETP Vol. 11, p. 573, 1960.
33. Bridgman, P. W. "Linear Compression to 30,000 Kg/cm² Including Relatively Incompressible Substances." Proceeding of the American Academy of Arts and Sciences, Vol. 77, p. 187, 1948.
34. Truesdell, C. "General and Exact Theory of Waves in Finite Elastic Strain." Arch. Rational Mech. Anal., Vol. 8, pp. 264-296, 1961.

35. Altshuler, L. V., Kormer, S. B., Brazhnik, M. I., Vladimirox, L. A., Speranskaya, M. P., and Funtikov, A. I. "The Isentropic Compressibility of Aluminum, Copper, Lead, and Iron at High Pressures." Soviet Physics, JETP, Vol. 11, No. 4, pp. 766-775, October 1960.
36. Malvern, L. E. "Plastic Wave Propagation in a Bar of Material Exhibiting a Strain-Rate Effect." Q. App. Math., Vol. 8, 1950-51, p. 405-.
37. Hill, R. The Mathematical Theory of Plasticity. Oxford University Press, 1950.
38. Dorn, J. E. and Hauser, F. Dislocation Concepts of Strain-Rate Effects. Symposium on Structural Dynamics under High Impulsive Loading, Dayton, Ohio. September 1962 (to be published).
39. Wood, D. S. "On Longitudinal Plane Waves of Elastic-Plastic Strain in Solids." J. App. Mech., Vol. 19, 1952, pp. 521-525.
40. Bell, J. F. "Propagation of Large Amplitude Waves in Annealed Aluminum." J. of Applied Physics, Vol. 31, 1960, pp. 277-282.
41. Bell, J. F. and Werner, W. M. "Applicability of the Taylor Theory of the Polycrystalline Aggregate to Finite Amplitude Wave Propagation in Annealed Copper." J. of Applied Physics, Vol. 33, 1962, pp. 2416-2425.
42. Sperrazza, J. Propagation of Large Amplitude Waves in Pure Lead. Proceedings Fourth U.S. National Congress of Applied Mechanics, Berkeley, California, June 1962 (to be published).

REFERENCES Continued

43. Johnson, P. C., Stein, B. A., and Davis, R. S. Basic Parameters of Metal Behavior under High Rate Forming. Arthur D. Little, Inc. Report to Watertown Arsenal Laboratories. WAL-TR-11.2/20, 1961.
44. Perzyna, P. The Study of the Dynamic Behavior of Rate Sensitive Plastic Materials. Brown University, Report to Office of Naval Research NR-064-406. Tech. Rept. No. 77, 1962
45. Perzyna, P. The Constitutive Equations for Rate Sensitive Plastic Materials. Brown University, Report to Office of Naval Research. NR-064-406. Tech. Rept. No. 76, 1962.
46. McQueen, R. G. and Marsh, S. P. One-Dimensional Compression Waves in Isotropic Solids. Los Alamos Scientific Laboratory (unpublished report, 1961).
47. Hauser, F. E., Simmons, J. A., and Dorn, J. E. Strain Rate Effects in Plastic Wave Propagation. Materials Research Laboratory, University of California, Berkeley, Technical Report, 1960.
48. Tapley, B. D. Stress-Strain Characteristics of Materials at High Strain Rates. Part VI. The Propagation of Plastic Waves in Finite Cylinders of Strain-Rate-Dependent Material. University of Texas, Report to the Sandia Corporation, 1960.
49. Fowles, G. R. Shock Wave Compression of Hardened and Annealed 2024 Aluminum. AFOSR No. 117, Stanford Research Institute, 1960.

REFERENCES Concluded

50. Lundergan, C. D. The Hugoniot Equation of State of 6061-T6 Aluminum at Low Pressures. Sandia Corporation, Albuquerque, New Mexico, Report No. SC-4637(RR), 1961.
51. Dorn, J. E., Goldberg, A., and Tietz, T. E. "The Effect of Thermal-Mechanical History on the Strain Hardening of Metals." Trans. Amer. Inst. Min. Metall. Engineers, (Institute of Metals), Vol. 180, 1949, pp. 205-224.
52. Herrmann, W. and Mack, E. Wave II Fortran Program for the Calculation of One-Dimensional Wave Propagation. Massachusetts Institute of Technology, Aeroelastic and Structures Research Laboratory Report, ASRL-1005, Dec. 1962.
53. Lundergan, C. D. Private Communication.

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